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Secția I

**MECANICĂ
MATEMATICĂ**

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Secția I

**MECANICĂ
MATEMATICĂ**

1986

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Secția I

MECANICĂ. MATEMATICĂ

S U M A R

I. ENESCU, Asupra unei algebre neasociative de ordin șase (franc., rez. rom.)	1
MIRCEA ȘTEFANOVICI, Algebra ternară atașată ecuației lui Rayleigh (rusă, rez. rom.)	5
VERONICA T. BORCEA și AL. NEAGU, Derivate scalare și aplicații cvasiconforme în varietăți riemanniene (engl., rez. rom.)	9
GH. ANDRICIOAEI, Asupra unui spațiu Einstein Finsler generalizat (engl., rez. rom.)	15
TEMISTOCLE BÎRSAN, Cîteva proprietăți ale spațiilor T -metrice (franc., rez. rom.)	19
NICOLETA NEGOESCU, Asupra unor teoreme de puncte fixe comune pentru multifuncții (franc., rez. rom.)	25
ROBERT T. VESCAN, Sisteme de inegalități de evoluție de obstacol implicit neregulat (engl., rez. rom.)	31
D. LEPĂDATU, Scurtă privire asupra rezolvării automate a problemelor de optimizare geometrică cu structură particulară (engl., rez. rom.)	37
CĂTĂLINA LOVINESCU, Flux în matroizi cu elemente paralele (engl., rez. rom.)	43
CORNELIU D. CIUBOTARIU și VIOREL A. STANCU, Moduri instabile într-un sistem fascicul-plasmă (engl., rez. rom.)	47
VICTOR-FLORIN PÔTERAȘU și N. MIHALACHE, Metoda elementelor de frontieră în problemele de torsiune elastică folosind teorema generalizată a lui Green și funcțiile R (engl., rez. rom.)	53
C. ACKER, Asupra invarianței axei instantanee de rotație a rigidului (III) (franc., rez. rom.)	61
ELISABETA RUSU, Problemă de termoelasticitate cuplată pentru un mediu infinit cu gol-cilindric (engl., rez. rom.)	65
SILVIA GABRIELA CIUMAȘU, Șoc termic pe frontiera unui semispațiu micropolar elăstic (engl., rez. rom.)	73
D. MANOLE, O teoremă de reciprocitate și o teoremă variațională pentru medii elastice cu microstructură (engl., rez. rom.)	79
ANASTASIA PĂDURARU, TRAIAN VIDRAȘCU, M. BARBU și EMIL LUCA, Modelarea matematică a dependenței dintre conținutul în ioni de Ca al feritei din sistemul $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ și unele proprietăți electrice (engl., rez. rom.)	85
SOFIA MELINTE, DAN MELINTE, ALEXANDRINA JEFLEA și MARCEL MOISE, Asupra unor caracteristici ale celulelor solare tip Schottky (engl., rez. rom.)	91
AL. DINCĂ, O nouă relație pentru $\cos \theta$ (franc., rez. rom.)	95
GEORGETA STRAT, C. GHIRVU, C. ONISCU, M. STRAT și V. NIȚICĂ, Influența solventului asupra spectrelor electronice de absorbție ale unor derivați ai acidului 4-amido-sulfonil-fenoxiacetic (engl., rez. rom.)	101

Section I

MECHANICS, MATHEMATICS

C O N T E N T S

I. ENESCU, Sur une algèbre nonassociative d'ordre six (French, Romanian summary)	1
MIRCEA ȘTEFANOVICI, Ternary Algebra Related to Rayleigh Equation (Russian, Romanian summary)	5
VERONICA T. BORCEA and AL. NEAGU, Scalar Derivatives and Quasiconformal Mappings in Riemannian Manifolds (English, Romanian summary)	9
GH. ANDRICIOAEI, On a Generalized Einstein-Finsler Space (English, Romanian summary)	15
TEMISTOCLE BÎRSAN, Quelques propriétés des espaces T -métriques (French, Romanian summary)	19
NICOLETA NEGOESCU, Sur des théorèmes de points fixes communs pour des fonctions multivoques (French, Romanian summary)	25
ROBERT T. VESCAN, Systems of Evolution Inequalities with Irregular Implicit Obstacle (English, Romanian summary)	31
D. LEPĂDATU, Short Survey on an Automatic Solving of Geometrical Optimization Problems with Particular Structure (English, Romanian summary)	37
CĂTĂLINA LOVINESCU, Flow in Matroids with Parallel Elements (English, Romanian summary)	43
CORNELIU D. CIUBOTARIU and VIOREL A. STANCU, Unstable Modes on a Beam-Plasma System (English, Romanian summary)	47
VICTOR-FLORIN POTERAȘU and N. MIHALACHE, B.E.M. in Elastic Torsion Problems using Green's Generalized Theorem and R -functions (English, Romanian summary)	53
C. ACKER, Sur l'invariance de l'axe instantané de rotation (III) (French, Romanian summary)	61
ELISABETA RUSU, Coupled Thermoelastic Problem for an Infinite Medium with a Cylindrical Cavity (English, Romanian summary)	65
SILVIA GABRIELA CIUMAȘU, Thermal Shock on the Boundary of Elastic Micropolar Half Space (English, Romanian summary)	73
D. MANOLE, A Reciprocity Theorem and a Variational Theorem for Elastic Media with Microstructure (English, Romanian summary)	79
ANASTASIA PĂDURARU, TRAIAN VIDRAȘCU, M. BARBU and EMIL LUCA, Mathematical Modelling of the Relation between the Ca Ion Content of Ferrite in the $\text{Ca}_2\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ System and some Electrical Properties (English, Romanian summary)	85
SOFIA MELINTE, DAN MELINTE, ALEXANDRINA JEFLEA and MARCEL MOISE, On some Characteristics of the Schottky Solar Cells (English, Romanian summary)	91
AL. DINCĂ, Une nouvelle relation pour $\cos \theta$ (French, Romanian summary)	95
GEORGETA STRAT, C. GHIRVU, C. ONISCU, M. STRAT and V. NIȚICĂ, Influence of Solvent upon the Absorption Electronic Spectra of some Derivatives of 4-amido-sulphonyl-phenoxyacetic Acid (English, Romanian summary)	101

Секция I

МЕХАНИКА. МАТЕМАТИКА

СО Д Е Р Ж А Н И Е

И. ЭНЕСКУ, Об одном неассоциативной алгебре шестого порядка (франц., рез. рум.)	1
МИРЧА ШТЕФАНОВИЧ, Тернарная алгебра, ассоциированная с уравнением Рэлея (русски, рез. рум.)	5
ВЕРОНИКА Т. БОРЧА и АЛ. НЯГУ, Скалярные производные и квазиконформные отображения в римановых многообразиях (англ., рез. рум.)	9
Г. АНДРИЧИОАЕИ, Об обобщенного Финслер-Эйнштейна пространства (англ., рез. рум.)	15
ТЕМИСТОКЛЕ БЫРСАН, Некоторые свойства T -метрических пространств (франц., рез. рум.)	19
НИКОЛЕТА НЕГОЕЗСКУ, О некоторых теоремах неподвижной точки, общей для многозначных функций (франц., рез. рум.)	25
РОБЕРТ Т. ВЕСКАН, Системы эволюционных неравенств с иррегулярными неявными ограничениями (англ., рез. рум.)	31
Д. ЛЕПЭДАТУ, Вкратце об автоматической задач геометрической оптимизации с частной структурой (англ., рез. рум.)	37
КЭТЭЛИНА ЛОВИНЕСКУ, Поток в матроидах с параллельными элементами (англ., рез. рум.)	43
КОРНЕЛЮ Д. ЧЮБОТАРЮ и ВИОРЕЛ СТАНКУ, Неустойчивые моды в системе плазма-пучок (англ., рез. рум.)	47
ВИКТОР-ФЛОРИН ПОТЕРАШУ и Н. МИХАЛАКЕ, Метод граничных элементов о задачах упругого кручения с применением обобщенной теоремы Грина и R -функции (англ., рез. рум.)	53
К. АККЕР, Об инвариантности мгновенной оси вращения твёрдого тела (III) (франц., рез. рум.)	61
ЭЛИСАБЕТА РУСУ, Задачи термоупругости для бесконечной среды с цилиндрической полостью (англ., рез. рум.)	65
СИЛВИА ГАБРИЕЛА ЧЮМАШУ, Термический удар на границе микрополиарного упругого полупространства (англ., рез. рум.)	73
Д. МАНОЛЕ, Теоремы взаимности и вариационные теоремы в теории линейной вязкоупругости с микроструктурой (англ., рез. рум.)	79
АНАСТАСИЯ ПЭДУРАРУ, ТРАЯН ВИДРАШКУ, М. БАРЕБУ и ЭМИЛ ЛУКА, Математическое моделирование зависимости между содержанием ионов Са феррита из систем $\text{Ca}_2\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ и некоторые электрические свойства (англ., рез. рум.)	85
СОФИЯ МЕЛИНТЕ, ДАН МЕЛИНТЕ, АЛЕКСАНДРИНА ЖЕФЛЯ и МАРЧЕЛ МОЙСЕ, О характеристиках солнечных элементов типа Шоттки (англ., рез. рум.)	91
АЛ. ДИНКЭ, Новое отношение для $\cos \theta$ (франц., рез. рум.)	95
ДЖЕОРДЖЕТА СТРАТ, К. ГИРВУ, К. ОНИСКУ, М. СТРАТ и В. НИЦИКЭ, Влияние растворителя на электронные спектры поглощения производных 4-амидосульфонилафенокси уксусной кислоты (англ., рез. рум.)	101

SUR UNE ALGÈBRE NONASSOCIATIVE D'ORDRE SIX

PAR

I. ENESCU

1. Soit $x = (\alpha_1, \alpha_2, \alpha_3)$, $y = (\beta_1, \beta_2, \beta_3)$ deux éléments quelconques de l'ensemble $\mathcal{A}$ des ternes ordonnées de nombres complexes. Par les opérations

$$(*) \quad x + y = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3), \quad \lambda x = (\lambda \alpha_1, \lambda \alpha_2, \lambda \alpha_3);$$

$$(1) \quad xy = (\alpha_2 \beta_3 + \alpha_3 \beta_2, \alpha_1 \beta_3 + \alpha_3 \beta_1, \alpha_1 \beta_2 + \alpha_2 \beta_1),$$

$$\forall \lambda \in \mathbb{R}, \quad \forall \alpha_j \in \mathbb{C}, \quad \forall \beta_j \in \mathbb{C}, \quad j = \overline{1, 3},$$

$\mathcal{A}$ devient une algèbre d'ordre six, à scalaires réels, commutative mais nonassociative. Une base dans $\mathcal{A}$ est formée par les éléments

$$(2) \quad \begin{aligned} u_1 &= (1, 0, 0), & u_2 &= (0, 1, 0), & u_3 &= (0, 0, 1), \\ u_4 &= (i, 0, 0), & u_5 &= (0, i, 0), & u_6 &= (0, 0, i), \end{aligned}$$

où $i^2 = -1$. Le calcul des produits $u_j u_k$ selon (1), conduit, pour la base (2), à la table de multiplication de l'algèbre $\mathcal{A}$, à savoir

$$(3) \quad \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 0 & u_3 & u_2 & 0 & u_6 & u_5 \\ u_2 & u_3 & 0 & u_1 & u_6 & 0 & u_4 \\ u_3 & u_2 & u_1 & 0 & u_5 & u_4 & 0 \\ u_4 & 0 & u_6 & u_5 & 0 & -u_3 & -u_2 \\ u_5 & u_6 & 0 & u_4 & -u_3 & 0 & -u_1 \\ u_6 & u_5 & u_4 & 0 & -u_2 & -u_1 & 0 \end{array}$$

Maintenant, considérons l'algèbre $\mathcal{A}_3$ donnée par la table

$$(**) \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & e_3 & e_2 \\ e_2 & e_3 & 0 & e_1 \\ e_3 & e_2 & e_1 & 0 \end{array}$$

étudiée dans [1] et, respectivement l'algèbre $\mathbb{C}$ des nombres complexes donnée par $\varepsilon_1^2 = \varepsilon_1$, $\varepsilon_1 \varepsilon_2 = \varepsilon_2 \varepsilon_1 = \varepsilon_2$, $\varepsilon_2^2 = -\varepsilon_1$.

En posant : $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$; $\varepsilon_1 = (1, 0)$, $\varepsilon_2 = (0, 1)$,

nous allons construire le produit tensoriel $\mathcal{A}_3 \otimes \mathbb{C}$, pour lequel une base est donnée par

$$p_1 = \varepsilon_1 e_1, \quad p_2 = \varepsilon_1 e_2, \quad p_3 = \varepsilon_1 e_3, \quad p_4 = \varepsilon_2 e_1, \quad p_5 = \varepsilon_2 e_2, \quad p_6 = \varepsilon_2 e_3.$$

Le calcul des produits $p_j p_k$, $j, k = \overline{1, 6}$, conduit à une table identique à (3) en démontrant le

Théorème 1. *L'algèbre $\mathcal{A}$, d'ordre six, définie par la multiplication (1), est le produit tensoriel de l'algèbre $\mathcal{A}_3$ par le corps des nombres complexes.*

Remarque. D'une certaine manière, ce résultat était prévisible, par ce que le produit (1) est le même par lequel est définie la multiplication des ternes ordonnés des nombres réels dans l'algèbre $\mathcal{A}_3$.

2. Nous allons étudier la structure de l'algèbre nonassociative $\mathcal{A}$, donnée par la table (3).

Tout d'abord, remarquons directement de la table (3), que l'algèbre $\mathcal{A}_3$ est contenue dans $\mathcal{A}$, en étant représentée par la sousalgèbre générée par les éléments u_1, u_2, u_3 . Existente, donc dans $\mathcal{A}$, au moins, quatre éléments idempotentes, à savoir ceux de l'algèbre $\mathcal{A}_3$, ayant respectivement les expressions [1]

$$(4) \quad \begin{aligned} E_1 &= \frac{1}{2}(u_1 + u_2 + u_3), & E_2 &= -\frac{1}{2}(-u_1 + u_2 + u_3), \\ E_3 &= -\frac{1}{2}(u_1 - u_2 + u_3), & E_4 &= \frac{1}{2}(-u_1 - u_2 + u_3). \end{aligned}$$

Soit $x = A_1 u_1 + A_2 u_2 + A_3 u_3 + A_4 u_4 + A_5 u_5 + A_6 u_6$, un élément arbitraire de $\mathcal{A}$. Le produit $E_1 x$, effectué compte tenu de (3), conduit à

$$E_1 x = \frac{1}{2} [A_2 + A_3] u_1 + (A_1 + A_3) u_2 + (A_1 + A_2) u_3 + (A_5 + A_6) u_4 + (A_4 + A_6) u_5 + (A_4 + A_5) u_6].$$

Enfin, en calculant la somme $x + 2xE_1$, on aboutit à l'identité

$$(5) \quad x + 2xE_1 = 2(A_1 + A_2 + A_3)E_1 + (A_4 + A_5 + A_6)(u_4 + u_5 + u_6),$$

valable pour tout élément $x \in \mathcal{A}$.

Directement par calcul, on constate que

$$(6) \quad E_1(u_4 + u_5 + u_6) = u_4 + u_5 + u_6.$$

Nous allons maintenant étudier l'existence dans $\mathcal{A}$ des idéaux propres et, soit $\mathcal{J}$ un tel idéal.

I. $E_1 \in \mathcal{J}$. Alors, selon (6) il en suit $(u_4 + u_5 + u_6) \in \mathcal{J}$ et, compte tenu de (5), il en résulte que, tout élément $x \in \mathcal{A}$ est contenu dans $\mathcal{J}$, fait qui conduit à la conclusion que $\mathcal{A} = \mathcal{J}$.

II. $E_1 \notin \mathcal{J}$ et, soit x arbitraire dans $\mathcal{J}$. Alors $(x + 2xE_1) \in \mathcal{J}$; nous allons démontrer que, si $E_1 \notin \mathcal{J}$ alors $(u_4 + u_5 + u_6) \notin \mathcal{J}$. En effet, supposons que $(u_4 + u_5 + u_6) \in \mathcal{J}$, ce qui implique $(A_4 + A_5 + A_6)(u_4 + u_5 + u_6) \in \mathcal{J}$, avec A_4, A_5, A_6 scalaires réels arbitraires. En conséquence, il en suit que $x + 2xE_1 - (A_4 + A_5 + A_6)(u_4 + u_5 + u_6) = 2(A_1 + A_2 + A_3)E_1 \in \mathcal{J}$, évidemment impossible. En posant $w_1 = 2(A_1 + A_2 + A_3)E_1$, $w_2 = (A_4 + A_5 + A_6)(u_4 + u_5 + u_6)$ et, compte tenu de (5), il en suit que la somme $w_1 + w_2$ est contenue

dans $\mathcal{J}$ bien que $w_1 \notin \mathcal{J}$, $w_2 \notin \mathcal{J}$. Mais, si $(w_1 + w_2) \in \mathcal{J}$, il en suit que le produit $(w_1 - w_2)(w_1 + w_2) = w_1^2 - w_2^2$ est aussi dans $\mathcal{J}$. Directement, par calcul, on constate que $w_1^2 - w_2^2$ est un multiple scalaire de w_1 , donc, il n'est pas contenu dans $\mathcal{J}$. Cette contradiction démontre que l'hypothèse $E_1 \notin \mathcal{J}$ est fausse, donc $E_1 \in \mathcal{J}$, en établissant ainsi le

Théorème 2. *L'algèbre $\mathcal{A} = \mathcal{A}_3 \otimes \mathbb{C}$ est simple.*

3. Nous allons maintenant étudier certaines propriétés des sousalgèbres de l'algèbre $\mathcal{A}$. Il existe de telles sousalgèbres, ainsi comme on voit directement de la table (3) : l'algèbre $\mathcal{A}_3$ et les sousalgèbres données par

$$(7) \quad \begin{array}{c|ccc} & u_3 & u_4 & u_5 \\ \hline u_3 & 0 & u_5 & u_4 \\ u_4 & u_5 & 0 & -u_3 \\ u_5 & u_4 - u_3 & 0 & \end{array}, \quad \begin{array}{c|ccc} & u_1 & u_5 & u_6 \\ \hline u_1 & 0 & u_6 & u_5 \\ u_5 & u_6 & 0 & -u_1 \\ u_6 & u_5 - u_1 & 0 & \end{array}, \quad \begin{array}{c|ccc} & u_2 & u_4 & u_6 \\ \hline u_2 & 0 & u_6 & u_4 \\ u_4 & u_6 & 0 & -u_2 \\ u_6 & u_4 - u_2 & 0 & \end{array}$$

Les trois sousalgèbres (7) sont isomorphes comme on voit facilement. Mais, elles ne sont pas isomorphes à la sousalgèbre $\mathcal{A}_3$ par ce que, aucune sousalgèbre (7) n'a pas des idempotents différents de zero, pendant que $\mathcal{A}_3$ possède de tels éléments.

Soit $\mathcal{K}$ une sousalgèbre propre de $\mathcal{A}$ et $\mathcal{J}$ un idéal dans $\mathcal{K}$. Nous supposons que $E_1 \in \mathcal{K}$.

1) $E_1 \in \mathcal{J}$. En notant par x un élément arbitraire de $\mathcal{K}$ et compte tenu de (5) et (6), il en résulte $x \in \mathcal{J}$, donc $\mathcal{K} = \mathcal{J}$.

2) $E_1 \notin \mathcal{J}$ et soit x arbitraire dans $\mathcal{J}$. Nous remarquons que, si $(u_4 + u_5 + u_6) \in \mathcal{J}$, alors, compte tenu de (5), $E_1 \in \mathcal{J}$, donc l'hypothèse $E_1 \notin \mathcal{J}$ implique $(u_4 + u_5 + u_6) \notin \mathcal{J}$. D'autre part, comme $E_1 \in \mathcal{K}$, il en résulte $xE_1 \in \mathcal{K}$ et, alors, quelque soit $x \in \mathcal{K}$, les éléments w_1 et w_2 sont, tous les deux, dans $\mathcal{K}$. Si $w_1 + w_2$ est dans $\mathcal{J}$, ainsi comme résulte ayant en vue l'identité (5) et compte tenu que x est arbitraire dans $\mathcal{J}$, il faut que le produit $(w_1 + w_2)(w_1 - w_2) = w_1^2 - w_2^2$ soit, aussi, dans $\mathcal{J}$. Mais $w_1^2 - w_2^2$ est de la forme λw_1 , $\lambda \in \mathbb{R}$, et, en conséquence, n'est pas contenu dans $\mathcal{J}$. Nous avons ainsi démontré le

Théorème 3. *Toute sousalgèbre $\mathcal{K}$ qui contient l'idempotent E_1 est simple.*

Cherchons maintenant, dans l'algèbre $\mathcal{A}$, l'ensemble des solutions de l'équation $E_1 x = x$, où $x = \sum_{j=1}^6 A_j u_j$, $A_j \in \mathbb{R}$. On aboutit au système d'équations

$$\begin{aligned} A_2 + A_3 &= 2A_1, & A_1 + A_3 &= 2A_2, & A_1 + A_2 &= 2A_3, \\ A_5 + A_6 &= 2A_4, & A_4 + A_6 &= 2A_5, & A_4 + A_5 &= 2A_6, \end{aligned}$$

qui a seulement des solutions de la forme: $A_1 = A_2 = A_3 = \lambda$; $A_4 = A_5 = A_6 = \mu$, λ, μ nombres réels arbitraires. Donc, l'ensemble des solutions de l'équation $E_1 x = x$ est formée avec tous les éléments $x \in \mathcal{A}$, de la forme

$$x = \lambda(u_1 + u_2 + u_3) + \mu(u_4 + u_5 + u_6), \quad \forall \lambda \in \mathbb{R}, \quad \forall \mu \in \mathbb{R}.$$

Remarque. Pour $\lambda = \frac{1}{2}$ et $\mu = 0$ on obtient $x = E_1$.

Comme les éléments $v_1 = u_1 + u_2 + u_3$, $v_2 = u_4 + u_5 + u_6$ sont linéaires indépendants, soit $h_1 = \frac{1}{2}v_1$, $h_2 = -\frac{1}{2}v_2$. L'ensemble des éléments $\{\lambda h_1 + \mu h_2\}$

forme une algèbre réelle d'ordre deux pour laquelle h_1, h_2 constituent une base avec le produit donné par $h_1^2 = h_1$, $h_1 h_2 = h_2 h_1 = h_2$, $h_2^2 = -h_1$, ce qui démontre le

Théorème 4. *L'ensemble des solutions de l'équation $E_1 x = x$ forme le corps complexe qui est contenu, comme sousalgèbre propre dans $\mathcal{A}$.*

Remarque. Si l'élément E_1 n'est pas contenu dans une sousalgèbre $\mathcal{X}$, alors les éléments h_1, h_2 ne sont pas contenus dans $\mathcal{X}$. En effet, $h_1 = \frac{1}{2}E_1$ et si $h_2 \in \mathcal{X}$ alors $h_2^2 = -h_1 \in \mathcal{X}$ ce qui est impossible. Donc, toute sousalgèbre qui ne contient l'élément E_1 est formée (en exceptant $x=0$), seulement par des éléments x caractérisés par la condition $E_1 x \neq x$.

Il existe de telles sousalgèbres, comme on voit de (7). De plus, les sousalgèbres (7) sont simples. En effet, soit $\mathcal{X}_1$ la sousalgèbre (7) qui a la base formée par les éléments u_1, u_5, u_6 . Directement, par calcul, on constate que $\mathcal{X}_1$ n'a pas des idempotents différents de zéro et les nilpotents d'indice deux sont $\{\lambda u_j\}$, $j=1, 5, 6$ et $\lambda \in \mathbb{R}$. Soit $\mathcal{I}$ un idéal de $\mathcal{X}_1$.

1) L'ordre de $\mathcal{I}=1$ et, alors, $\mathcal{I} \approx \mathbb{R}$ ou $\mathcal{I}$ est l'algèbre nulle d'ordre un. Le cas $\mathcal{I} \approx \mathbb{R}$ n'est pas possible, par ce que $\mathcal{X}_1$ n'a pas aucun idempotent différent de zéro. Si $\mathcal{I}$ est l'algèbre nulle d'ordre un, un nilpotent u_j , j fixé, est contenu dans $\mathcal{I}$. Mais comme $u_j u_k = u_s$, $j \neq k \neq i \neq j$, il en résulte que $u_j u_k \notin \mathcal{I}$, donc l'algèbre nulle d'ordre un n'est pas idéal dans $\mathcal{X}_1$.

2) L'ordre de $\mathcal{I}=2$. S'il existe un élément u_j , j fixé, alors $\mathcal{I}$ contient aussi le produit $u_j u_k = u_s$, $j, k, s=1, 5, 6$, donc $\mathcal{I} = \mathcal{X}_1$. Si $\mathcal{I}$ ne contient aucun nilpotent u_j , soit $x = A_1 u_1 + A_5 u_5 + A_6 u_6$ arbitraire dans $\mathcal{I}$. Le produit $(xu_1)^2 = -2A_5 A_6 u_1$ et, comme $(xu_1) \in \mathcal{I}$, il en résulte $u_1 \in \mathcal{I}$ ce qui est impossible, donc $\mathcal{X}_1 = \mathcal{I}$.

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BIBLIOGRAPHIE

1. Enescu I., *Sur une algèbre réelle nonassociative*. Bul. Inst. polit. Iași, XXVII (XXXI), 1-2, s. I, 9-12 (1981).

ASUPRA UNEI ALGEBRE NEASOCIATIVE DE ORDIN ȘASE

(Rezumat)

Se consideră mulțimea ternelor ordonate de numere complexe care se structurează prin operațiile (*) și (1). Se obține astfel o algebră $\mathcal{A}$, reală, de ordinul șase, comutativă dar neasociativă dată, în baza (2), prin tabela (3).

Se arată că algebra $\mathcal{A}$ este produsul tensorial dintre algebra $\mathcal{A}_3$, dată prin tabela (**), studiată în [1] și corpul $\mathbb{C}$ al numerelor complexe.

Se arată că $\mathcal{A}$ este o algebră simplă și se stabilesc unele proprietăți ale subalgebrelor sale.

ТЕРНАРНАЯ АЛГЕБРА, АССОЦИИРОВАННАЯ С УРАВНЕНИЕМ РЭЛЕЯ

МИРЧА ШТЕФАНОВИЧ

В теории нелинейных колебаний известно уравнение Рэля [1] для малых нелинейных амортизаций

$$(1) \quad \ddot{x} + x + \varepsilon(m\dot{x}^3 - n\dot{x}) = 0, \quad m, n > 0, \quad 0 < \varepsilon < 1.$$

Уравнение (1) эквивалентно системе

$$(2) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= \varepsilon(ny - my^3) - x. \end{aligned}$$

Решениями системы (2) являются решения, расположенные в плоскости $z = 1$ системы кубических дифференциальных уравнений

$$(3) \quad \begin{aligned} \dot{x} &= yz^2, \\ \dot{y} &= n\varepsilon yz^2 - m\varepsilon y^3 - xz^2, \\ z &= 0. \end{aligned}$$

Неассоциативная вещественная тернарная алгебра, ассоциированная [2] системе (3), является алгеброй с базисом e_1, e_2, e_3 и следующей таблицей умножения:

$$(4) \quad \begin{aligned} \overline{e_1 e_3 e_3} &= \overline{e_3 e_1 e_3} = \overline{e_3 e_3 e_1} = -\frac{1}{3} e_2, \\ \overline{e_2 e_3 e_3} &= \overline{e_3 e_2 e_3} = \overline{e_3 e_3 e_2} = \frac{1}{3} e_1 + \frac{1}{3} n\varepsilon e_2, \\ \overline{e_2 e_2 e_2} &= -m\varepsilon e_2, \end{aligned}$$

остальные произведения $e_i e_j e_k$ равны нулю.

Алгебра, заданная таблицей (4), изоморфна, в результате изменения базиса

$$u_1 = \frac{1}{n\varepsilon\sqrt{m\varepsilon}} e_1, \quad u_2 = \frac{1}{\sqrt{m\varepsilon}} e_2, \quad u_3 = \sqrt{\frac{3}{n\varepsilon}} e_3,$$

алгебре

$$(5) \quad \begin{aligned} \overline{u_1 u_3 u_3} &= \overline{u_3 u_1 u_3} = \overline{u_3 u_3 u_1} = -\frac{1}{n^2 \varepsilon^2} u_2, \\ \overline{u_2 u_3 u_3} &= \overline{u_3 u_2 u_3} = \overline{u_3 u_3 u_2} = u_1 + u_2, \\ \overline{u_2 u_2 u_2} &= -u_2, \end{aligned}$$

остальные произведения $\overline{u_i u_j u_k}$ равны нулю.

Система кубических дифференциальных уравнений, соответствующая алгебре (5) будет

$$(6) \quad \begin{cases} \dot{u} = 3vw^2, \\ \dot{v} = 3vw^2 - v^3 - \frac{3}{n^2 \epsilon^2} uw^2, \\ \dot{w} = 0. \end{cases}$$

Система (6) аффинно эквивалентна системе (3) в результате следующего преобразования координат :

$$(7) \quad x = \frac{1}{n\epsilon\sqrt{m\epsilon}} u, \quad y = \frac{1}{\sqrt{m\epsilon}} v, \quad z = \sqrt{\frac{3}{m\epsilon}} w.$$

Посмотрим сколько неизоморфных типов алгебр (5) существует для всех значений параметра $a = 1/(n^2 \epsilon^2) > 0$.

Предположим, что существует линейная функция φ , которая изоморфно отображает алгебру (5), соответствующую значению a , на алгебру (5), соответствующую другому значению $b \neq a$. Пусть φ дана соотношениями

$$(8) \quad \begin{cases} \varphi(u_1) = Au_1 + Bu_2 + Cu_3, \\ \varphi(u_2) = Du_1 + Eu_2 + Fu_3, \\ \varphi(u_3) = Gu_1 + Hu_2 + Iu_3, \end{cases}$$

где матрица коэффициентов $\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$ должна быть невырожденной.

Из условия

$$\overline{\varphi(u_2) \varphi(u_2) \varphi(u_2)} = -\varphi(u_2),$$

следует $F=0$, $D=0$ и $E^3=E$.

Так как E не может быть равен нулю, то $E^2=1$. При найденных значениях из условия

$$\overline{\varphi(u_2) \varphi(u_3) \varphi(u_3)} = \overline{\varphi(u_3) \varphi(u_2) \varphi(u_3)} = \overline{\varphi(u_3) \varphi(u_3) \varphi(u_2)} = \varphi(u_1) + \varphi(u_2)$$

следует $C=0$ (значит $I \neq 0$), $A = \pm I^2$, $\pm(I^2 - H^2) = B \pm 1$. Тогда из

$$\overline{\varphi(u_1) \varphi(u_1) \varphi(u_1)} = 0$$

следует $B=0$, из

$$\overline{\varphi(u_2) \varphi(u_2) \varphi(u_3)} = \overline{\varphi(u_2) \varphi(u_3) \varphi(u_2)} = \overline{\varphi(u_3) \varphi(u_2) \varphi(u_2)} = 0$$

следует $H=0$, следовательно $I^2=1$, $A^2=1$. Из

$$\overline{\varphi(u_3) \varphi(u_3) \varphi(u_3)} = 0$$

следует $G=0$, а из

$$\overline{\varphi(u_1) \varphi(u_3) \varphi(u_3)} = \overline{\varphi(u_3) \varphi(u_1) \varphi(u_3)} = \overline{\varphi(u_3) \varphi(u_3) \varphi(u_1)} = -b\varphi(u_2),$$

следует $a=b$, что противоречит гипотезе.

Таким образом, существует бесконечное число неизоморфных алгебр (5), а также бесконечное число аффинно неэквивалентных уравнений (1), которые соответствуют различным значениям параметра $n\varepsilon > 0$.

В алгебре (5) существуют только два нильпотентных элемента u_1 и u_3 ($u_1u_1u_1 = u_3u_3u_3 = 0$) следовательно, полупрямые через начало координат, направленные по векторам $\pm u_1$ и $\pm u_3$ состоят из стационарных точек системы (6). Алгебра (5) не содержит идемпотенты, но содержит антиидемпотент u_2 ($u_2u_2u_2 = -u_2$) значит прямая, направленная по вектору u_2 , проходящая через начало координат, является решением системы (6).

Алгебра (5) неизоморфна алгебре, ассоциированной с уравнением Ван дер Поля [2], которая не содержит антиидемпотенты.

Ясно, что алгебра (5) имеет собственный идеал, порожденный элементами u_1 и u_2 .

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ЛИТЕРАТУРА

1. Buzdugan Gh., Fetcu L., Radeș M., *Vibrațiile sistemelor mecanice*. Ed. Acad. Buc., 1975.
2. Ștefanovici M., *Cubic Differential Equations and Nonassociative Ternary Algebras*. Bul. Inst. polit. Iași, supliment „Cercetări matematice”, 1985, pp. 21–24.

ALGEBRA TERNARĂ ATAȘATĂ ECUAȚIEI LUI RAYLEIGH

(Rezumat)

Se studiază algebra ternară neasociativă (4) atașată ecuației lui Rayleigh (1). Se arată că există o infinitate de tipuri neizomorfe de algebre (4) și deci o infinitate de ecuații (1) neechivalente afin, corespunzătoare diferitelor valori ale parametrului $n\varepsilon > 0$.

SCALAR DERIVATIVES AND QUASICONFORMAL MAPPINGS IN RIEMANNIAN MANIFOLDS

BY

VERONICA T. BORCEA and AL. NEAGU

The purpose of this paper is a characterisation of quasiconformal mappings in a Riemannian manifold in terms of scalar derivatives. The equivalence with the Gehring's metric definition is proved and the mappings with bounded triangular dilatation and θ -mappings are investigated.

Let V_n be a Riemannian manifold with the metric tensor $\dot{g}$ and the geodesic metric d . We denote by W the neighbourhood of $0_x \in T_x V_n$ where the exponential map $\exp_x$ is a diffeomorphism. We define the upper and the lower scalar derivative of a map $f: U \subset V_n \rightarrow V_n$ by

$$(1) \quad D^+f(x) = \limsup_{y \rightarrow x} \frac{d(f(x), f(y))}{d(x, y)}, \quad D^-f(x) = \liminf_{y \rightarrow x} \frac{d(f(x), f(y))}{d(x, y)}.$$

We have generally $0 \leq D^-f(x) \leq D^+f(x) < \infty$. If the scalar derivatives of f at x are simultaneously equal to 0 or ∞ we say that f is degenerate at x . Because $\exp_x$ has a behaviour as an isometry for the distances measured from 0_x it follows that

$$(2) \quad \begin{aligned} d(x, y) &= d(\exp_x^{-1}(x), \exp_x^{-1}(y)) = \|\exp_x^{-1}(y)\|, \\ d(f(x), f(y)) &= \|\exp_{f(x)}^{-1}(f(y))\|. \end{aligned}$$

Proposition 1. *The scalar derivatives of the map $f: U \subset V_n \rightarrow V_n$ are given by*

$$(3) \quad D^+f(x) = D^+F_x(0_x) \quad \text{and} \quad D^-f(x) = D^-F_x(0_x),$$

where $F = \exp_{f(x)}^{-1} \circ f \circ \exp_x$.

Proof. Let us consider $Y \in T_x V_n$ and $y = \exp_x Y$, then $F_x(Y) = \exp_{f(x)}^{-1}(f(y))$ and $F_x(0_x) = 0_{f(x)}$. It follows that $d(f(x), f(y)) = \|F_x(Y)\| = d(F_x(0_x), F_x(Y))$ and $d(x, y) = \|\exp_x^{-1}(y)\| = \|Y\| = d(0_x, Y)$.

Proposition 2. *If f is differentiable at x then*

$$(4) \quad D^+f(x) = \|Df(x)\| \quad \text{and} \quad (5) \quad D^-f(x) = \inf_{Y \in T_x V_n \setminus \{0_x\}} \{\|Df(x)Y\| / \|Y\|\}.$$

Proof. By the definition of $D^+f(x)$, (3) and [3] it follows that

$$\begin{aligned} D^+f(x) &= D^+F_x(0_x) = \sup_{Y \in T_x V_n \setminus \{0_x\}} \{\|DF_x(0_x)Y\| / \|Y\|\} = \\ &= \sup_{Y \in T_x V_n \setminus \{0_x\}} \{\|Df(x)Y\| / \|Y\|\} = \|Df(x)\|. \end{aligned}$$

Remark 1. If $J_f(x) \neq 0$ then $D^-f(x) = 1 / \|Df(x)^{-1}\|$, where $J_f(x)$ is the Jacobian of f at x .

Proposition 3. If f is differentiable at x , then

$$(6) \quad D^+f(x) = \sqrt{\lambda^+} \quad \text{and} \quad (7) \quad D^-f(x) = \sqrt{\lambda^-},$$

where λ^+ and λ^- are respectively the largest and the smallest eigenvalue of the operator $(Df(x))^* Df(x)$, $((Df(x))^*$ is the adjoint operator).

Proof. From $Df(x) = DF_x(0_x)$ it follows that $(Df(x))^* Df(x) = (DF_x(0_x))^* DF_x(0_x)$ and by [3] $D^+f(x) = D^+F_x(0_x) = \sqrt{\lambda^+}$; $D^-f(x) = D^-F_x(0_x) = \sqrt{\lambda^-}$.

Proposition 4. If the map f is homeomorphic in a neighbourhood of x , then

$$(8) \quad D^+f^{-1}(x) = (D^-f(x))^{-1} \quad \text{and} \quad (9) \quad D^-f^{-1}(x^*) = (D^+f(x))^{-1},$$

where $x^* = f(x)$.

Proof. If f is a homeomorphism it follows that F_x is also a homeomorphism and, by Proposition 1 and [3] we have (8) and (9).

Let us consider a map $f: D \subset V_n \rightarrow D^* \subset V_n$, where D is a domain in V_n . The geodesic dilatation of f at $x \in D$ is, by definition

$$\delta(x) = \limsup_{r \rightarrow 0} \frac{L(x, r)}{l(x, r)},$$

where $L(x, r) = \sup_y \{d(f(x), f(y)) : d(x, y) = r\}$; $l(x, r) = \inf \{d(f(x), f(y)) : d(x, y) = r\}$.

A homeomorphism $f: D \subset V_n \rightarrow D^* \subset V_n$ is called K -quasiconformal (K -qc) by the Gehring's metric definition ($1 \leq K < \infty$) if $\delta(x) \leq K$, for every $x \in D$.

Theorem 1. If f is nondegenerate at x then

$$(10) \quad \delta(x) \leq \frac{D^+f(x)}{D^-f(x)}.$$

Proof. We can assume that $D^-f(x) > 0$ and $D^+f(x) < \infty$. Let $\varepsilon > 0$ be such that $D^-f(x) > \varepsilon$. By (1) it follows that there exists $r_0 > 0$ such that

$$(11) \quad D^-f(x) - \varepsilon < \frac{d(f(x), f(y))}{d(x, y)} < D^+f(x) + \varepsilon,$$

whenever $0 < d(x, y) < r_0$. For $0 < r < r_0$ and by (11) we infer

$$r(D^-f(x) - \varepsilon) \leq \inf_{d(x,y)=r} \{d(f(x), f(y))\} \leq \sup_{d(x,y)=r} \{d(f(x), f(y))\} \leq r(D^+f(x) + \varepsilon).$$

Whence

$$(12) \quad \frac{\sup_{d(x,y)=r} \{d(f(x), f(y))\}}{\inf_{d(x,y)=r} \{d(f(x), f(y))\}} \leq \frac{D^+f(x) + \varepsilon}{D^-f(x) - \varepsilon}.$$

Consequently

$$\delta(x) = \limsup_{r \rightarrow 0} \frac{\sup_{d(x,y)=r} \{d(f(x), f(y))\}}{\inf_{d(x,y)=r} \{d(f(x), f(y))\}} \leq \frac{D^+f(x) + \varepsilon}{D^-f(x) - \varepsilon}$$

and, as ε is arbitrary, we get (10).

Proposition 5. *The following relations hold:*

$$(13) \quad D^+f(x) = \limsup_{r \rightarrow 0} \frac{\sup_{d(x,y)=r} \{d(f(x), f(y))\}}{r},$$

$$(14) \quad D^-f(x) = \liminf_{r \rightarrow 0} \frac{\inf_{d(x,y)=r} \{d(f(x), f(y))\}}{r}.$$

Proof. For any y such that $d(x, y) = r$ we have

$$\frac{d(f(x), f(y))}{d(x, y)} \leq \frac{1}{r} \sup_{d(x,y)=r} \{d(f(x), f(y))\}, \text{ hence } D^+f(x) \leq \limsup_{r \rightarrow 0} \left[\frac{1}{r} \sup_{d(x,y)=r} \{d(f(x), f(y))\} \right].$$

In order to prove the opposite inequality, let $\varepsilon > 0$ be, then there exists $r_0 > 0$ such that $d(f(x), f(y))/d(x, y) < D^+f(x) + \varepsilon$, whenever $0 < d(x, y) < r_0$. We obtain

$$\frac{1}{r} \sup_{d(x,y)=r} \{d(f(x), f(y))\} \leq D^+f(x) + \varepsilon \quad \text{for } 0 < r < r_0.$$

By letting $r \rightarrow 0$ we get

$$\limsup_{r \rightarrow 0} \frac{\sup_{d(x,y)=r} \{d(f(x), f(y))\}}{r} \leq D^+f(x) + \varepsilon$$

and, as ε was arbitrary it follows the inequality. The relation (15) is proved analogously.

Theorem 2. *If f is a differentiable homeomorphism and nondegenerate at x then*

$$(15) \quad \delta(x) = \frac{D^+f(x)}{D^-f(x)}.$$

Proof. By (5) and (6) we have

$$(16) \quad D^+f(x) = \sup \{ \|Df(x) \cdot Y\|, \quad \|Y\| = 1 \},$$

$$(17) \quad D^-f(x) = \inf \{ \|Df(x) \cdot Y\|, \quad \|Y\| = 1 \}.$$

Let ε be an arbitrary positive number. Since $F(Y) = DF(0_x) \cdot Y + \alpha(Y)$, where $\lim_{Y \rightarrow 0_x} \frac{\alpha(Y)}{\|Y\|} = 0$, there exists a small enough number $r_0 > 0$ such that

$$\|Df(x) \cdot Y\| - \varepsilon \|Y\| \leq d(f(x), f(y)) \leq \|Df(x) \cdot Y\| + \varepsilon \|Y\|$$

whenever $0 < d(x, y) < r_0$, where $Df(x) = DF(0_x)$ and $d(f(x), f(y)) = \|F(Y)\|$. If $d(x, y) = r$ (hence $\|Y\| = r$), $0 < r < r_0$ we obtain

$$(18) \quad \|Df(x) \cdot Y\| - \varepsilon r \leq \sup_{d(x, y) = r} \{d(f(x), f(y))\} = L(x, r),$$

$$(19) \quad l(x, r) = \inf_{d(x, y) = r} \{d(f(x), f(y))\} \leq \|Df(x) \cdot Y\| + \varepsilon r.$$

It follows that

$$(20) \quad rD^+f(x) - \varepsilon r \leq L(x, r),$$

$$(21) \quad l(x, r) \leq rD^-f(x) + \varepsilon r,$$

whence

$$(22) \quad \frac{D^+f(x) - \varepsilon}{D^-f(x) + \varepsilon} \leq \frac{L(x, r)}{l(x, r)}, \quad 0 < r < r_0.$$

By letting $r \rightarrow 0$ and as ε was arbitrary, we get

$$(23) \quad \frac{D^+f(x)}{D^-f(x)} \leq \delta(x).$$

By (10) and (23) we obtain (15).

Definition 1. A homeomorphism $f: D \subset V_n \rightarrow D^* \subset V_n$ is K -qc in terms of scalar derivatives iff:

(i) f is nondegenerate at x , $x \in D$;

(ii) $D^+f(x) \leq KD^-f(x)$, for every $x \in D$.

Remark 2. A nondegenerate differentiable homeomorphism $f: D \rightarrow D^*$ is K -qc in terms of scalar derivatives iff it is K -qc according to Gehring's metric definition.

Definition 2. The mapping $f: D \subset V_n \rightarrow V_n$ is said to have bounded triangular dilatation if there exists a real number $C, C \geq 1$, such that for every $x, y, z \in D$ with $d(x, y) \leq d(y, z)$ we have $d(f(x), f(y)) \leq Cd(f(y), f(z))$.

Definition 3. The mapping $f: D \subset V_n \rightarrow V_n$ is said to have locally bounded triangular dilatation if, for every $x \in D$, there exists $r > 0$ such that $d(f(x), f(y)) \leq Cd(f(x), f(z))$ for every $y, z \in D$ with $d(y, x) \leq d(z, x) \leq r, C \geq 1, C \in \mathbb{R}$.

Proposition 6. Let $f: D \rightarrow D^*$ be a map such that $D^-f(x) > 0$ and $D^+f(x) < \infty$, $x \in D$. Then f has locally bounded triangular dilatation.

Proof. We assume, by absurd, that for every $C \geq 1$, there exists $u, v \in D$ and a small enough number $\varepsilon > 0$ with $d(u, x) \leq d(v, x) + \varepsilon$ such that $d(f(u), f(x)) > Cd(f(v), f(x))$. By letting $C = n$ and $\varepsilon = 1/n$ it follows that there exist $u_n, v_n \in D$ with $d(u_n, x) \leq d(v_n, x) \leq 1/n$ and $d(f(u_n), f(x)) > nd(f(v_n), f(x))$. Then

$$\frac{d(f(u_n), f(x))}{d(u_n, x)} > n \frac{d(f(v_n), f(x))}{d(u_n, x)} \geq n \frac{d(f(v_n), f(x))}{d(v_n, x)}$$

for every $n \in N$ and we have

$$\liminf_{n \rightarrow \infty} \frac{d(f(v_n), f(x))}{d(v_n, x)} \geq \liminf_{v \rightarrow x} \frac{d(f(v), f(x))}{d(v, x)} = D^- f(x) > 0,$$

$$\limsup_{n \rightarrow \infty} \frac{d(f(u_n), f(x))}{d(u_n, x)} \geq \limsup_{n \rightarrow \infty} n \frac{d(f(v_n), f(x))}{d(v_n, x)} = \infty,$$

hence

$$D^+ f(x) \geq \limsup_{n \rightarrow \infty} \frac{d(f(u_n), f(x))}{d(u_n, x)} \geq \infty \quad \text{contradicting the hypothesis.}$$

Theorem 3. A nondegenerate differentiable homeomorphism $f: D \rightarrow D^*$ has locally bounded triangular dilatation iff f is quasiconformal in D (in terms of scalar derivatives).

Proof. If f has locally bounded triangular dilatation the assertion follows by Theorem 4 [1] and Remark 2.

Conversely, we assume that f is quasiconformal in terms of scalar derivatives in D . Then, we have $D^- f(x) > 0$ and $D^+ f(x) < \infty$, $\forall x \in D$. By Proposition 6 it follows that f has locally bounded triangular dilatation.

Remark 3. If f is K -qc in D then the constant C from definition may be chosen $C = K + \varepsilon$, for every $\varepsilon > 0$.

Definition 3. A homeomorphism $f: D \rightarrow D^*$ (D, D^* domains in V_n) is said to be a θ -mapping if there exists a function θ which is continuous and strictly increasing in $[0, 1]$ with $\theta(0) = 0$, such that, if $x, y \in \Delta$ with $d(x, y) < d(x, \partial\Delta)$ then

$$\frac{d(f(x), f(y))}{d(f(x), \partial\Delta^*)} \leq \theta \frac{d(x, y)}{d(x, \partial\Delta)} \quad \text{for every subdomain } \Delta \subset D, \quad \Delta^* = f(\Delta).$$

Theorem 4. Let $f: D \rightarrow D^*$ be a θ -mapping. Then f has bounded triangular dilatation.

Proof. Let $x_0 \in D$ be such that $0 < r < \frac{1}{2} d(x_0, \partial D)$ and $x, y, z \in B(x_0, r/2)$ with $d(x, y) \leq d(y, z)$. Let γ be the geodesic which joins x with y and $r_1 = d(x, y)$. We consider $S(x, r_1/2) \cap \gamma = \{u, v\}$ and choose u such that $d(y, u) = r_1/2$. Then

$$d(u, v) = d(u, x) + d(v, x) = r_1, \quad d(y, v) = d(y, u) + d(u, v) = r_1/2 + r_1 = 3r_1/2,$$

$$d(x_0, v) \leq d(x_0, x) + d(x, v) \leq r/2 + d(x, y)/r \leq r \quad \text{hence} \quad u, v \in B(x_0, r).$$

Let be $\Delta = D \setminus \{v\}$ and $\Delta^* = D^* \setminus \{f(v)\}$. We have $d(u, \partial\Delta) = d(u, v)$, $d(x, u) = d(x, y)/2 = d(u, v)/2 = d(u, \partial\Delta)/2$, hence $d(x, u) \leq d(u, \partial\Delta)$ and $d(f(u), \partial\Delta^*) \leq d(f(u), f(v))$. It follows that

$$\frac{d(f(x), f(u))}{d(f(u), \partial\Delta^*)} \leq \theta \left[\frac{d(x, u)}{d(u, \partial\Delta)} \right] \quad \text{which implies}$$

$$(24) \quad d(f(x), f(u)) \leq \theta(\tfrac{1}{2}) d(f(u), f(v)).$$

In the same way we obtain

$$(25) \quad d(f(y), f(u)) \leq \theta(\tfrac{1}{2})d(f(u), f(v)).$$

From (24) and (25) we have

$$(26) \quad d(f(x), f(y)) \leq 2\theta(\tfrac{1}{2})d(f(u), f(v)).$$

Now let be $\Delta_1 = D \setminus \{z\}$, $\Delta_1^* = D^* \setminus \{f(z)\}$. We have $d(x, y) \leq d(y, z) = d(y, \partial\Delta_1)$, $d(y, u) = d(x, y)/2 \leq d(y, \partial\Delta_1)/2$, $d(f(y), \partial\Delta_1^*) \leq d(f(y), f(z))$. Hence

$$\frac{d(f(u), f(y))}{d(f(y), \partial\Delta_1^*)} \leq \theta \left[\frac{d(u, y)}{d(y, \partial\Delta_1)} \right] \quad \text{implies} \quad d(f(u), f(y)) \leq \theta(\tfrac{1}{2})d(f(y), f(z)).$$

Analogously we obtain

$$(27) \quad d(f(v), f(y)) \leq \theta(\tfrac{1}{2})d(f(y), f(z))$$

and hence

$$(28) \quad d(f(u), f(v)) \leq 2\theta(\tfrac{1}{2})d(f(y), f(z)).$$

The inequalities (26) and (28) imply $d(f(x), f(y)) \leq 4\theta^2(\tfrac{1}{2})d(f(y), f(z))$.

Theorem 5. *If f^{-1} is a θ -mapping then f has bounded triangular dilatation.*

Proof. Let $x_0 \in D$ be and we choose $r < d(x_0, \partial D)$ such that for every $x, y \in B(x_0, r)$, $d(f(x), f(y)) \leq d(f(y), \partial D^*)$. Let be $x, y, z \in B(x_0, r)$ with $d(x, y) \leq d(y, z)$ and $\Delta = D \setminus \{x\}$. We have $d(y, \partial\Delta) = d(x, y)$. Since $\Delta^* = D^* \setminus \{f(x)\}$ we have $d(f(x), \partial\Delta^*) \leq d(f(x), f(y))$ whence $d(f(x), \partial\Delta^*) = d(f(x), f(y))$. We may write

$$1 \leq \frac{d(z, y)}{d(x, y)} = \frac{d(f^{-1}(f(z)), f^{-1}(f(y)))}{d(f^{-1}(f(y)), \partial\Delta)} \leq \theta \left[\frac{d(f(z), f(y))}{d(f(y), \partial\Delta^*)} \right] = \theta \left[\frac{d(f(z), f(y))}{d(f(y), f(x))} \right].$$

As θ is increasing, we obtain $d(f(x), f(y)) \leq d(f(y), f(z))/\theta^{-1}(1)$ and the theorem is proved.

Corollary. *If f is a θ -mapping, then f and f^{-1} have the locally bounded triangular dilatation.*

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REFERENCES

1. Borcea V. T., *On Quasiconformal Mappings in Riemannian Manifolds of Dimension n* . An. șt. Univ. „Al. I. Cuza”, 21, 125–128 (1985).
2. Caraman P., *n -dimensional Quasiconformal Mappings*. Ed. Acad. R.S.R., București (Romania) and Abacus Press Tunbridge, Wells, Kent (England), 1974.
3. Frunză M., *Scalar Derivatives and Quasiconformal Mappings*. Rev. Roum. Math. Pures Appl., 24, 1175–1181 (1979).
4. Frunză M., *Contribuții la studiul reprezentărilor cvaziconforme*. Teză de doctorat, Buc., 1979.
5. Gheorghiev Gh., Oproiu V., *Geometrie diferențială*. Ed. did. și ped., Buc., 1977.

DERIVATE SCALARE ȘI APLICAȚII CVASICONFORME ÎN VARIETĂȚI RIEMANNIENE (Rezumat)

Se dă o caracterizare a aplicațiilor cvasiconforme în varietăți riemanniene cu ajutorul derivatelor scalare. Se demonstrează echivalența cu definiția metrică a lui Gehring și se studiază aplicațiile cu dilatare triunghiulară mărginită și θ -aplicațiile.

ON A GENERALIZED EINSTEIN-FINSLER SPACE

BY

GH. ANDRICIOAEI

1. Starting from an idea of A. Moór [1], S. Watanabe and F. Ikeda [4] obtain a generalization of the Finsler space, different of that obtained by R. Miron in [5], as it follows.

Let M be a n -dimensional differentiable manifold, TM its tangent bundle and $\pi : TM \rightarrow M$ the natural projection. If (U, x^i) is a locally chart on M , then $\pi^{-1}(U) \subset TM$ is a coordinate neighborhood and (x^i, y^i) are the coordinate of a point $y \in \pi^{-1}(U)$, $x = (x^i) = \pi(y)$. If

1) on $\pi^{-1}(U)$ is defined a symmetric and nondegenerate Finsler tensor field g of type $(0,2)$ with local components $g_{ij}(x, y)$;

2) the matrices

$$(1.1) \quad A_j^i = \delta_j^i + g^{tr} y^s \partial_j g_{rs},$$

$$(1.2) \quad A^{pq}_{kj} = 2\delta_k^q \delta_j^p + \delta_k^q g^{pr} y^i \partial_r g_{ij} + B_r^i C_{s00} g^{sp} g^{qr} \partial_i g_{jk} - g^{rp} y^i \delta_j^q \partial_r g_{ki},$$

where $C_{i00} = C_{ijk} y^j y^k = \frac{1}{2} \partial_i g_{jk} y^j y^k$, are inversable, having the inverse matrix B_j^i and B^{kj}_{rs} respectively, then there exists a Finsler connection of Cartan type denoted by

$$(1.3) \quad {}^*C\Gamma = (F_{jk}^i, N_k^i, C_{jk}^i),$$

which satisfies the conditions:

a) ${}^*C\Gamma$ is h - and v -metrical, i.e., $g_{ij|k} = 0$, $g_{ij}|_k = 0$;

b) the torsion tensors T and S^1 vanish i.e., $T_{jk}^i = F_{jk}^i - F_{kj}^i = 0$, $S_{jk}^i = C_{jk}^i - C_{kj}^i = 0$;

c) $F_{oj}^i = N_j^i$, where $F_{oj}^i = y^k F_{kj}^i$.

We shall call this (M, g) a generalized Finsler space in Watanabe and Ikeda sense.

2. In the following we propose to study the isotropy of this space after the model of Rund [2] and Matsumoto [3].

In terms of ${}^*C\Gamma$ we consider the curvature tensor of Rund

$$(2.1) \quad K_{hjk}^i = \delta_k F_{hj}^i - \delta_j F_{hk}^i + F_{hj}^r F_{rk}^i - F_{hk}^r F_{rj}^i, \quad K_{hijk} = g_{ir} K_{hjk}^r.$$

Also, we associate to the metrical structure g , the function

$$(2.2) \quad L(x, y) = g_{ij}(x, y) y^i y^j,$$

called the absolute energy (Miron [5]) and impose

$$(2.3) \quad L(x, y) > 0 \text{ for every } x \in M \text{ and } y \in TM. \text{ In this case } l^i = y^i / \sqrt{L} \text{ are the components of the unit vector of } y(y^i).$$

Next, we consider the sectional curvature $K(x, y, X)$ given by

$$(2.4) \quad K_{ijk} y^i x^j y^h X^k = K(x, y, X) (g_{ih} g_{jk} - g_{ik} g_{jh}) y^i X^j y^h X^k,$$

where y is the supporting element in $x(x^i) \in M$ and $X \in TM$.

Definition 1. A generalized Finsler space (M, g) in Watanabe and Ikeda sense is of scalar curvature (isotropic) if the invariant K from (2.4) is independent of X^i . In this case the curvature will be denoted by $R(x, y)$.

Let z_a^i ($a=1, \dots, n$) be an orthonormal linear frame at $x \in M$ and denote

$$(2.5) \quad M_{ab}(x, y) = K_{ijk} z_a^i z_b^j z_a^h z_b^k,$$

from which it results

$$(2.6) \quad M_a(x, y) = \sum_{b=1}^n M_{ab}(x, y) = K_{ijk} z_a^i z_a^j z_a^h z_a^k,$$

where we use the following relations

$$(2.7) \quad g_{ij} z_a^i z_b^j = \delta_{ab}, \quad \sum_b z_b^j z_b^k = g^{jk}, \quad K_{ih} = K_{i^j h^j} = g^{jk} K_{ijk}.$$

The quantity $M_a(x, y)$ which is constructed from z_a^i and does not depend on z_b^i ($b=1, \dots, n, b \neq a$) is called then mean curvature [2].

Taking the direction of z_a^i as that of y^i , i.e. $z_a^i = l^i = y^i / \sqrt{L}$, we obtain the mean curvature with respect to y^i in the form

$$(2.8) \quad M(x, y) = L^{-1} K_{hij} y^h y^j (= K_{hij} l^h l^j).$$

Now let us consider the equation

$$(2.9) \quad K_{hj}(x, y) y^h y^j = 1,$$

which for the fixed x gives a hypersurface of M_x which is called K -hypersurface at x .

Definition 2. A direction of the principal curvature at x is such that the radius of the K -hypersurface at x has an extremal length in that direction.

Thus, by using a Lagrange multiplier λ , a direction of principal curvature is obtained from the equation

$$(2.10) \quad \partial_k [g_{ij}(x, y) y^i y^j - \lambda (K_{ij} y^i y^j - 1)] = 0.$$

After performing the calculations, we obtain

$$(2.11) \quad \partial_k (g_{ij}) y^i y^j + 2g_{kj} y^j - \lambda [\partial_k (K_{ij}) y^i y^j + K_{kj} y^j + K_{jk} y^j] = 0.$$

Multiplying (2.11) with y_k and taking account of (2.8) we obtain λ which replaced into (2.11) gives

$$(2.12) \quad \frac{K_{o||o} + 2LM}{g_{o||o} + 2L} (g_{o||o} + 2g_{kj} y^j) = \partial_k (K_{ij}) y^i y^j + K_{kj} y^j + K_{jk} y^j,$$

where index o means contraction by y , i.e., $K_{o||o} = \partial_k (K_{ij}) y^i y^j y^k$.

Definition 3. A generalized Finsler space in Watanabe-Ikeda sense is called a generalized Einstein Finsler space if it is homogeneous in every point $x \in M$ with respect to the mean curvature, i.e., any direction y^i is a direction of the principal curvature.

Summarizing the above results, we have

Proposition 2.1. A generalized Finsler space in Watanabe-Ikeda sense is a generalized Einstein Finsler space if and only if there exists a Finsler scalar field $M(x, y)$ which satisfies the equation (2.12) for any y^i .

Now we shall prove the following

Theorem 2.1. A generalized Finsler space in Watanabe-Ikeda sense is a generalized Einstein Finsler space if and only if the equation

$$(2.13) \quad (K_{o||o} - M g_{o||o}) (g_{o||k} + 2g_{kj} y^j) = (n-1) L (g_{o||o} + 2L) \partial_k R(x, y)$$

holds. In this case the mean curvature is $M(x, y) = (n-1) R(x, y)$.

Proof. The equation (2.4) is rewritten in our case as

$$(2.14) \quad [R(x, y) (L g_{jk} - g_{ik} g_{jh} y^i y^h) - K_{ijhk} y^i y^h] X^j X^k = 0.$$

If we take account of Definition 1, then from (2.14) we have

$$2R(x, y) (L g_{jk} - g_{ik} g_{jh} y^i y^h) - (K_{ijhk} + K_{ikhj}) y^i y^h = 0.$$

Multiplying this equation by g^{jm} and contracting $m=k$, we obtain

$$(2.15) \quad (n-1) R(x, y) L(x, y) - K_{ih} y^i y^h = 0.$$

From (2.8) and (2.5) we have

$$(2.16) \quad (n-1) R(x, y) = M(x, y).$$

Differentiating with respect to y^k the relation (2.15) we obtain

$$(2.17) \quad (n-1) \partial_k R \cdot L + (n-1) R \partial_k L = \partial_k (K_{ih}) y^i y^h + K_{kh} y^h + K_{ik} y^i.$$

Taking into account (2.2) and (2.16), we finally have

$$(2.18) \quad (n-1) L \partial_k R + M [\partial_k (g_{ij}) y^i y^j + 2g_{kj} y^j] = \partial_k (K_{jh}) y^i y^h + K_{kh} y^h + K_{ik} y^i.$$

The assertion of the theorem is proved if we compare (2.18) with (2.12).

The integrability conditions of (2.13) are obtained, if we differentiate this equation with respect to y^i and equate the mixed partial derivatives. Thus it follows

$$(2.19) \quad \dot{\partial}_i \alpha(g_{oo||k} + 2y_k) - \dot{\partial}_k \alpha(g_{oo||i} + y_i) + 2\alpha(g_{ko||i} - g_{io||k}) = 0,$$

where

$$\alpha = (K_{oo||o} - Mg_{oo||o}) [(n-1) L(g_{oo||o} + 2L)]^{-1}.$$

Remark 2.1. If g_{ij} is $o(p)$ -homogeneous with respect to y^i , then K_{ij} is $o(p)$ -homogeneous also with respect to y^i and we have $K_{oo||o} = 0$. In this case (2.13) lead us to

$$(2.20) \quad \dot{\partial}_k R = 0.$$

Hence we have

Corollary 2.1. *A generalized Finsler space in Watanabe-Ikeda sense, in which g_{ij} is $o(p)$ -homogeneous with respect to y , is an Einstein Finsler space if and only if $R(x, y)$ does not depend on y .*

Remark 2.2. Corollary 2.1 holds good also in the case of a generalized Finsler space with Miron's metric [5].

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REFERENCES

1. MOÓR A., *Entwicklung einer Geometrie der allgemeiner metrischen Linienkementräume*. Acta Sci. Math., 17, 85–120 (1956).
2. RUND H., *Über Finslersche Räume mit speziellen Krümmungseigenschaften*. Monatshefte für Mathematik, 66, 241–251 (1962).
3. MATSUMOTO M., *Foundation of Finsler Geometry and Special Finsler Spaces*. Kyoto, 1977 (unpublished).
4. WATANABE S., IKEDA F., *On Metrical Finsler Connections of a Metrical Finsler Structure*. Tensor, 39, 37–41 (1983).
5. MIRON R., *Metrical Finsler Structure and Metrical Finsler Connections*. J. Math. Kyoto Univ., 23 (1983).

ASUPRA UNUI SPAȚIU EINSTEIN FINSLER GENERALIZAT

(Rezumat)

Se studiază izotropia unui spațiu Finsler generalizat în sens Watanabe-Ikeda și se caracterizează un spațiu Finsler Einstein generalizat. Rezultatul rămâne valabil în cazul unui spațiu Finsler generalizat cu metrică Miron.

QUELQUES PROPRIÉTÉS DES ESPACES T -MÉTRIQUES

PAR

TEMISTOCLE BÎRSAN

1. Introduction. Les espaces T -métriques sont des généralisations des espaces métriques : dans l'inégalité triangulaire, l'opération d'addition est remplacée par une fonction $(a, b) \rightarrow T(a, b)$, $a, b \in \mathbb{R}_+$, appelée dans ce qui suit t -fonction (c-à-d. fonction triangulaire).

Définition [1]. Une fonction $d : X \times X \rightarrow \mathbb{R}_+$ est une métrique sur X par rapport à la t -fonction $T : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (en bref, d est une T -métrique) si elle satisfait aux conditions suivantes :

- I) $d(x, y) = 0 \Leftrightarrow x = y$;
- II) $d(x, y) = d(y, x)$, pour tous $x, y \in X$;
- III) $d(x, z) \leq T(d(x, y), d(y, z))$ pour tous $x, y, z \in X$. $x \neq y \neq z \neq x$.

Le triplet (X, d, T) est appelé *espace T -métrique*.

Etant donné un espace T -métrique (X, d, T) , on désigne par $S(x, r)$ et U_ε les ensembles $\{y ; d(x, y) < r\}$ et $\{(x, y) ; d(x, y) < \varepsilon\}$, respectivement. En général, la famille d'ensembles $\{S(x, r) ; r > 0\}$ (resp. $\{U_\varepsilon ; \varepsilon > 0\}$) ne forme pas un système fondamental de voisinages de x (resp. d'entourages) pour une topologie (resp. uniformité) sur X . Dans ce sens les propriétés de la t -fonction T sont décisives.

2. Métrisation des espaces T -métriques. On dit que l'espace (X, d, T) est métrisable s'il existe une métrique sur X qui a pour un système fondamental d'entourages l'ensembles de parties $\{U_\varepsilon ; \varepsilon > 0\}$. De même, une topologie sur X est compatible avec la T -métrique d si l'ensemble $\{S(x, r) ; r > 0\}$ est un système fondamental de voisinages de x , quel que soit $x \in X$. Si une telle topologie existe, elle est unique et séparée T_1 .

Théorème 1. Soit (X, d, T) un espace T -métrique dont la t -fonction T satisfait à la condition

$$(1) \quad \lim_{a \rightarrow 0, b \rightarrow 0} T(a, b) = 0.$$

Alors l'espace donné est métrisable.

Démonstration. Étant donné $n \in \mathbb{N}^*$ il y a, grâce à (1), $m \in \mathbb{N}$, $m > n$, tel que pour tous $a, b \in \mathbb{R}$, $0 < a < 1/m$ et $0 < b < 1/m$, on a $T(a, b) < 1/n$. Alors, on vérifie aisément que $U_m^2 \subset U_n$, où $U_k = \{(x, y) ; d(x, y) < 1/k\}$. Par suite, $\{U_n ; n \in \mathbb{N}^*\}$ est un système fondamental dénombrable d'entourages et, donc, l'espace (X, d, T) est pseudo-métrisable.

Mais l'espace donné est aussi séparé Hausdorff. En effet, si $x \neq y$, on a $d(x, y) = r > 0$ et on peut indiquer $r_1 > 0$ de sorte que $a, b \in \mathbb{R}$, $0 < a < r_1$ et $0 < b < r_1$ entraînent la relation $T(a, b) < r$. Alors, en utilisant l'inégalité triangulaire III, on prouve que les sphères $S(x, r_1)$ et $S(y, r_1)$ sont disjointes. En conclusion, l'espace (X, d, T) est métrisable.

Remarques. 1. Les hypothèses du Théorème 1 [1] impliquent (1). 2. La condition (1) n'est pas suffisante pour que les sphères ouvertes soient des ensembles ouverts (voir l'exemple ci-dessous). On trouve l'explication dans le fait que (1) est une condition locale.

Exemple. Considérons les ensembles $M_1 = \{(x, y) ; x \geq 0, y \geq 0\}$, $M_2 = \{(x, y) ; x < 0, y \geq 0\}$, $X = M_1 \cup M_2$ et deux fonctions $f, g : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$. Définissons $d : X \times X \rightarrow \mathbb{R}$ par

$$d(u, v) = \begin{cases} d_0(u, v) & u, v \in M_1 \text{ ou } u, v \in M_2, \\ f[d_0(u, 0) + d_0(v, 0)], & u \in M_i, v \in M_j, i \neq j (i, j = 1, 2), \end{cases}$$

où d_0 est la distance euclidienne, et la t -fonction $T : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ par

$$T(a, b) = \begin{cases} \alpha & a = b = 0, \\ g(a + b), & (a, b) \neq (0, 0), \end{cases}$$

ou $\alpha \in \mathbb{R}_+$.

Si les fonctions f et g satisfont aux conditions : (1) $g(t) \geq f(t) > t$, pour tout $t > 0$, et (2) $g(t) \geq g(t')$ dès que $t > t' > 0$, alors (X, d, T) est un espace T -métrique. Si g satisfait en plus à la condition $\lim_{t \rightarrow 0} g(t) = 0$, alors le Théorème 1 s'applique et l'espace est métrisable.

En prenant $\alpha = 0$, $g(t) = 2t$, pour tout $t > 0$, et

$$f(t) = \begin{cases} t, & t > 0, t \text{ rationnel}, \\ \varphi(t), & t > 0, t \text{ irrationnel}, \end{cases}$$

ou $\varphi : \mathbb{R}_+^* \rightarrow \mathbb{R}$ est donnée par

$$\varphi(t) = \begin{cases} 2t, & t > 1, \\ \left(1 + \frac{1}{2}\right)t, & \frac{1}{2} < t \leq 1, \\ \left(1 + \frac{1}{n}\right)t, & \frac{1}{n} < t \leq \frac{1}{n-1}, \quad n \geq 2, \end{cases}$$

les conditions 1—3 sont vérifiées. L'espace (X, d, T) est métrisable, T est une fonction continue et strictement croissante, mais il y a des sphères qui ne sont pas ouvertes. En effet, pour la sphère de centre $O(0, 0)$ et rayon 1 on a $S(0, 1) = \{(x, y) \in M_1 ; (x^2 + y^2)^{\frac{1}{2}} < 1\} \cup \{(x, y) \in M_2 ; (x^2 + y^2)^{\frac{1}{2}} \text{ rationnel et}$

$<1\} \cup \left\{ (x, y) \in M_2; (x^2 + y^2)^{\frac{1}{2}} \text{ irrationnel et } < \frac{2}{3} \right\}$. On vérifie sans peine que, par exemple, le point $\left(\frac{3}{4}, 0\right)$ appartient à $S(0,1)$ sans être point intérieur.

Théorème 2. Soit (X, d, T) un espace T -métrique dont la t -fonction T satisfait aux conditions :

1° $T(0, b) \leq b$, pour tout $b \in \mathbb{R}_+$;

2° $t \mapsto T(t, b)$ est une application scs (c-à-d. semi-continue supérieurement) au point $t=0$, quel que soit $b \in \mathbb{R}_+$. Alors :

1) il existe une topologie sur X compatible avec d ,

2) les sphères $S(x, r)$ sont des ensembles ouverts par rapport à elle.

Démonstration. Il est évident que : a) $x \in S(x, r)$ pour tout $r > 0$ et b) $S(x, r_1) \cap S(x, r_2) = S(x, r_0)$, ou $r_0 = \min(r_1, r_2)$. On établit facilement que c) pour tout nombre $r > 0$ et tout point $y \in S(x, r)$ il y a $r_1 > 0$ tel que $S(y, r_1) \subset S(x, r)$. En effet, si $d_0 = d(x, y)$ on a $T(0, d_0) \leq d_0 < r$. Grâce à l'hypothèse 2° il y a $r_1 > 0$ de sorte que $T(t, d_0) < r$ pour tout $t < r_1$. Par conséquent, si $z \in S(y, r_1)$ on a $d(y, z) < r_1$ et on obtient

$$d(x, z) = d(z, x) \leq T(d(z, y), d(y, x)) = T(d(z, y), d_0) < r,$$

c-à-d. $z \in S(x, r)$ et l'affirmation c) est vraie. Les propriétés a)–c) garantissent la justesse des deux conclusions du théorème.

En renforçant les hypothèses du Théorème 2, on obtient les résultats (dont la démonstration revient à la vérification de (1)) :

Théorème 3. Soit (X, d, T) un espace T -métrique de sorte que

1° $T(0, b) \leq b$, pour tout $b \in \mathbb{R}_+$, et

2° $t \mapsto T(t, b)$ est une application scs au point $t=0$ uniformément par rapport à $b \in K$, quel que soit l'ensemble compact $K \subset \mathbb{R}_+$. Alors l'espace (X, d, T) est métrisable.

Théorème 4. Soit (X, d, T) un espace T -métrique de sorte que

1° $T(0, b) \leq b$, pour tout $b \in \mathbb{R}_+$,

2° $T(a, b) \leq T(a, b')$ si $b' \geq b > 0$, et

3° $t \mapsto T(t, b)$ est scs au point $t=0$, quel que soit $b \in \mathbb{R}_+$. Alors l'espace (X, d, T) est métrisable.

Remarques. 1) La condition 2° du Théorème 3 est satisfaite si T est continue. 2) Dans les hypothèses du Théorème 3 (ou 4), les sphères $S(x, r)$ sont ouvertes. 3) Il est possible que la condition (1) ou les hypothèses du Théorème 1 [1] soient vérifiées, tandis que celles des Théorèmes 3 et 4 ne le sont pas. L'Exemple nous offre un tel cas ($g(t) = t$ entraîne que $T(0, b) = 2b > b$).

Le théorème suivant se démontre facilement.

Théorème 5. Soit (X, d, T) un espace T -métrique dont T a les propriétés :

1° $T(a, a) \leq a$, quel que soit $a \in \mathbb{R}_+$,

2° $T(a, b) < T(a', b')$ si $a' > a > 0$ et $b' > b > 0$.

Alors l'espace donné est ultramétrisable.

3. **Theorèmes de point fixe.** Étant donné un espace T -métrique (X, d, T) , une φ -contraction est une application $f: X \rightarrow X$ qui satisfait à la condition

$$(2) \quad d(fx, fy) \leq \varphi(d(x, y)), \quad x, y \in X,$$

où la fonction $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ est scs et $\varphi(t) < t$ si $t > 0$. Pour $\varphi(t) = qt$, ou $q \in \mathbb{R}$ et $0 < q < 1$, l'application f est une contraction. D. W. Boyd et J. S. W. Wong donnent dans ([2], Théorème 1) une extension du théorème de Picard-Banach au cas des φ -contractions. Nous allons obtenir un résultat analogue pour les espaces T -métriques.

Théorème 6. Soient (X, d, T) un espace T -métrique complet et $f: X \rightarrow X$ une φ -contraction. Si T vérifie les hypothèses du Théorème 3, alors :

1) f a un point fixe ξ unique ;

2) pour tout $x \in X$, la suite des approximations successives $(f^n x)_n$ converge vers ξ .

Démonstration. Soit $x \in X$ quelconque. Considérons la suite $(c_n)_n$ définie par $c_n = d(f^n x, f^{n-1} x)$. Il suit de (2) que $c_{n+1} \leq \varphi(c_n)$, et, donc, $c_{n+1} < c_n$, $n \in \mathbb{N}$. La suite étant strictement décroissante, il existe $c = \lim c_n$.

Si $c > 0$, alors $\varphi(c) < c$, et, pour n suffisamment grand, on a $\varphi(c_n) < c$. Par suite, $c < c_{n+1} \leq \varphi(c_n) < c$. Absurde. Donc $c = 0$.

La suite $(f^n x)_n$ est une suite de Cauchy, car dans le cas contraire, il y a un nombre $\varepsilon_0 > 0$ et les suites de nombres entiers $(m(k))_k$ et $(n(k))_k$ telles que

$$(3) \quad m(k) > n(k) \geq k, \quad k \in \mathbb{N}, \quad \text{et}$$

$$(4) \quad d_k = d(f^{m(k)} x, f^{n(k)} x) \geq \varepsilon_0, \quad k \in \mathbb{N}.$$

Choisissons $m(k)$ le plus petit entier supérieur à $n(k)$ pour lequel (4) a lieu.

Alors, on a

$$(5) \quad d(f^{m-1} x, f^n x) < \varepsilon_0.$$

Soient $\alpha, \varepsilon \in \mathbb{R}$, $\alpha > \varepsilon_0$ et $\varepsilon > 0$. Dans les hypothèses du théorème il y a $\delta > 0$, dépendant seulement de ε , tel que

$$(6) \quad T(t, b) < b + \varepsilon, \quad \text{pour tous } t < \delta \text{ et } b \in [0, \alpha].$$

La suite $(c_n)_n$ étant convergente à zéro, il y a $k_0 \in \mathbb{N}$ tel que $c_k < \delta$ quel que soit $k \geq k_0$. Alors, en vertu de (4)...(6) et de la monotonie de $(c_n)_n$, on a

$$\begin{aligned} d_k &\leq T(d(f^{m(k)} x, f^{m(k)-1} x), d(f^{m(k)-1} x, f^{n(k)} x)) = T(c_m, d(f^{m(k)-1} x, f^{n(k)} x)) < \\ &< d(f^{m(k)-1} x, f^{n(k)} x) + \varepsilon < \varepsilon_0 + \varepsilon, \end{aligned}$$

pour tout $k \geq k_0$. Par conséquent, $\varepsilon_0 \leq d_k < \varepsilon_0 + \varepsilon$, si $k \geq k_0$, d'où

$$(7) \quad \lim_k d_k = \varepsilon_0.$$

Considérons maintenant que $\varepsilon < 1/2 (\alpha - \varepsilon_0)$. De (2) et (7) et grâce aux propriétés de φ , on obtient

$$(8) \quad d(f^{n+1}x, f^{m+1}x) \leq \varphi(d_k) < \varepsilon_0 < \alpha, \quad k \geq k_0,$$

k_0 pouvant être choisi le même comme ci-dessus.

Mais, du fait que

$$d(f^n x, f^{m+1} x) \leq T(d(f^n x, f^{n+1} x), d(f^{n+1} x, f^{m+1} x)) = T(c_{n+1}, d(f^{n+1} x, f^{m+1} x))$$

et tenant compte de (6), (8) et $c_{n+1} < c_k < \delta$ si $k \geq k_0$, il résulte que

$$(9) \quad d(f^n x, f^{m+1} x) < d(f^{n+1} x, f^{m+1} x) + \varepsilon,$$

ou, en utilisant de nouveau (8),

$$(10) \quad d(f^n x, f^{m+1} x) < \varepsilon_0 + \varepsilon < \alpha, \quad k \geq k_0.$$

En vertu de (6), (10) et (9), on obtient

$$\begin{aligned} d_k &= d(f^m x, f^n x) \leq T(d(f^m x, f^{m+1} x), d(f^{m+1} x, f^n x)) = \\ &= T(c_{m+1}, d(f^{m+1} x, f^n x)) < d(f^{m+1} x, f^n x) + \varepsilon, \end{aligned}$$

donc

$$d_k < d(f^{n+1} x, f^{m+1} x) + 2\varepsilon.$$

D'ici et du fait que f est une φ -contraction, on a

$$d_k < \varphi(d_k) + 2\varepsilon, \quad k \geq k_0,$$

c-à-d. $\lim_k (d_k - \varphi(d_k)) = 0$. Tenant compte de (7), il résulte que $\lim_k (d_k) = \varepsilon_0$,

ce qui est en contradiction avec les propriétés de φ . Par suite, $(f^n x)_n$ est une suite de Cauchy, donc elle est convergente. On démontre habituellement que $\xi = \lim_n f^n x$ est le point fixe unique de f . Ce qu'il fallait démontrer.

Remarque. Si f est une contraction et les hypothèses du Théorème 6 relativement à l'espace (X, d, T) sont satisfaites, alors l'application f a un point fixe unique qui peut être obtenu par la méthode des approximations successives.

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BIBLIOGRAPHIE

1. Bîrsan T., Une application d'un théorème de J. Aczél. Publ. Math. (Debrecen), **31**, 113–118 (1984).
2. Boyd D. W., Wong J. S. W., On Nonlinear Contractions. Proc. Amer. Math. Soc., **20**, 458–464 (1969).

CÎTEVA PROPRIETĂȚI ALE SPAȚIILOR T -METRICE

(Rezumat)

Se dau câteva teoreme de metrizare a spațiilor T -metrice și o teoremă de punct fix pentru φ -contracții în cadrul acestor spații.

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SUR DES THÉORÈMES DE POINTS FIXES COMMUNS POUR DES FONCTIONS MULTIVOQUES

PAR

NICOLETA NEGOESCU

Dans cette Note nous avons résolu un problème ouvert, proposé par K. Y a n a g i, c'est à dire nous avons substitué la condition $a + (a+3)(b+c) < 1$, $a, b, c \geq 0$ par la condition relaxée $a + 2b + 2c < 1$, $a, b, c \geq 0$ dans les théorèmes de points fixes de S. I t o h [2] et K. Y a n a g i [4].

1. Soit (X, d) un espace métrique. Pour tous $x \in X$ et $A \subset X$ nous définissons $D(x, A) = \inf\{d(x, y) : y \in A\}$.

Soient $CB(X)$ la famille de sous ensembles nonvides, bornés et fermés de X .

Pour tous $A, B \subset X$ nous notons par $H(A, B)$ la distance Hausdorff-Pompeiu induite par la métrique d

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\}, \quad \forall A, B \in CB(X).$$

(X, d) est un espace métrique convexe si pour tous $x, y \in X$ il existe $z \in X$, $z \neq x$, $z \neq y$, tel que $d(x, y) = d(x, z) + d(z, y)$.

Les lemmes suivants sont des conséquences directes des définitions de la métrique Hausdorff-Pompeiu et pour $D(x, A)$.

L e m m a 1. Si $H(A, B) = 0$, alors $D(x, B) = D(y, A) = 0$ pour tous $x \in A$ et $y \in B$. Aussi nous avons évidemment $D(x, A) \leq d(x, y)$, $\forall y \in A$.

L e m m a 2. Soient $A, B \in CB(X)$ et $k \in (1, \infty)$ donné. Alors pour tout $a \in A$, il existe $b \in B$ tel que $d(a, b) \leq kH(A, B)$.

L e m m a 3. Si K est un sousensemble nonvide d'un espace métrique complet et convexe (X, d) , alors pour tous $x \in K$, et $y \notin K$ il existe un $z \in \partial K$ (la frontière de K) tel que $d(x, z) + d(z, y) = d(x, y)$.

Soit (X, d) un espace métrique complet et métrique convexe, K un sousensemble nonvide fermé de (X, d) et $T_n : K \rightarrow CB(X)$, $n \in \mathbb{N}^*$, une suite de fonctions multivoques et si les conditions suivantes sont satisfaites :

(1) ils existent les nombres réels $a, b, c > 0$ tels que $a + 2b + 2c < 1$;

(2) $H(T_i(x), T_j(y)) \leq ad(x, y) + b\{D(x, T_i(x)) + D(y, T_j(y))\} + c\{D(x, T_j(y)) + D(y, T_i(x))\}$ pour $j = i + 1$, $j \in \mathbb{N}^*$;

(3) $T_n(x) \in K$ pour tout $x \in K$ et tout $n \in \mathbb{N}^*$,

alors sont vraies les propositions suivantes :

Proposition 1. Si $a, b, c > 0$, alors ils existent $k > 1$ tel que $k(a+2b+2c) < 1$, c'est à dire $(ka+kb+kc)/(1-kb-kc) < 1$, et les suites (x_n) et (y_n) telles que pour tout $n \in \mathbb{N}$ nous avons :

$$(a) \quad y_{n+1} \in T_{n+1}(x_n),$$

$$(b) \quad d(y_n, y_{n+1}) \leq kH(T_n(x_{n-1}), T_{n+1}(x_n)) \text{ où}$$

$$(b_1) \text{ si } y_{n+1} \in K, \text{ alors nous prenons } x_{n+1} = y_{n+1} \text{ et}$$

$$(b_2) \text{ si } y_{n+1} \notin K, \text{ alors nous notons } x_{n+1} \in \partial K,$$

qui satisfait à l'inégalité $d(x_n, x_{n+1}) + d(x_{n+1}, y_n) = d(x_n, y_{n+1})$.

En effet, soit $x_1 = y_1 \in T_1(x_0)$ où $x_0 \in K$. Du Lemme 2 il existe $y_2 \in T_2(x_1)$ tel que $d(y_1, y_2) \leq kH(T_1(x_0), T_2(x_1))$. Si $y_2 \in K$, nous posons $x_2 = y_2$. Si $y_2 \notin K$, alors nous choisissons un élément $x_2 \in \partial K$ tel que $d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2)$, conformément au Lemme 3.

En continuant ce procédé, nous obtenons les suites (x_n) et (y_n) telles que pour $\forall n \in \mathbb{N}$ nous avons les propriétés (a), (b).

Proposition 2. Si $x_n = y_n$ et $x_{n+1} = y_{n+1}$, alors $d(x_n, x_{n+1}) \leq \{(ka+kb+kc)/(1-kb-kc)\} d(x_{n-1}, x_n)$.

Les condition (b) (Proposition 1), l'inégalité (2) et la définition de $D(\cdot, A)$ impliquent

$$\begin{aligned} d(x_n, x_{n+1}) &= d(y_n, y_{n+1}) \leq kH(T_n(x_{n-1}), T_{n+1}(x_n)) \leq \\ &\leq kad(x_{n-1}, x_n) + kb\{D(x_{n-1}, T_n(x_{n-1})) + D(x_n, T_{n+1}(x_n)) + \\ &+ kc\{D(x_{n-1}, T_{n+1}(x_n)) + D(x_n, T_n(x_{n-1}))\} \leq kad(x_{n-1}, x_n) + \\ &+ kb\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} + kc\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}. \end{aligned}$$

Donc $(1-kb-kc)d(x_n, x_{n+1}) \leq (ka+kb+kc)d(x_{n-1}, x_n)$ et

$$d(x_n, x_{n+1}) \leq \frac{ka+kb+kc}{1-kb-kc} d(x_{n-1}, x_n).$$

Proposition 3. Si $x_n = y_n$ et $x_{n+1} \neq y_{n+1}$, alors il suit

$$d(x_n, y_{n+1}) \leq \frac{ka+kb+kc}{1-kb-kc} d(x_{n-1}, x_n).$$

De (b_2) nous obtenons que $d(x_n, x_{n+1}) \leq d(x_n, y_{n+1}) = d(y_n, y_{n+1})$. De la même manière que dans la Proposition 2 nous obtenons

$$d(y_n, y_{n+1}) \leq \frac{ka+kb+kc}{1-kb-kc} d(x_{n-1}, x_n) \quad \text{et} \quad d(x_n, y_{n+1}) \leq \frac{ka+kb+kc}{1-kb-kc} d(x_{n-1}, x_n).$$

Proposition 4. Si $x_n \neq y_n$ et $x_{n+1} = y_{n+1}$, alors nous avons $x_{n-1} = y_{n-1}$ et $d(y_n, x_{n+1}) \leq \frac{ka+kb+kc}{1-kb-kc} d(x_{n-1}, y_n)$.

En effet, en appliquant les Lemmes 2, 1 et (2) nous obtenons

$$\begin{aligned} d(y_n, x_{n+1}) &\leq kH(T_n(x_{n-1}), T_{n+1}(x_n)) \leq kad(x_{n-1}, x_n) + \\ &+ kb\{D(x_{n-1}, T_n(x_n)) + D(x_n, T_{n+1}(x_n))\} + kc\{D(x_{n-1}, T_{n+1}(x_n)) + \\ &+ D(x_n, T_n(x_{n-1}))\} \leq kad(x_{n-1}, y_n) + kb\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} + \end{aligned}$$

+ $kc\{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\}$ et, en employant l'inégalité du triangle pour les distances $d(x_n, x_{n+1})$ et $d(x_{n-1}, x_{n+1})$ où nous introduisons y_n , il suit que :

$$d(y_n, x_{n+1}) \leq kad(x_{n-1}, y_n) + kb\{d(x_{n-1}, y_n) + d(y_n, x_{n+1})\} + kc\{d(x_{n-1}, y_n) + d(y_n, x_{n+1})\} \text{ ou } d(y_n, x_{n+1}) \leq kad(x_{n-1}, y_n) + kb\{d(x_{n-1}, y_n) + d(y_n, x_{n+1})\} + kc\{d(x_{n-1}, y_n) + d(y_n, x_{n+1})\}.$$

Donc $(1 - kb - kc)d(y_n, x_{n+1}) \leq kad(x_{n-1}, y_n) + k(b + c)d(x_{n-1}, y_n)$ ou $(1 - kb - kc)d(y_n, x_{n+1}) \leq (ka + kb + kc)d(x_{n-1}, y_n)$.

Conséquence 1. Pour tout $n \in \mathbb{N}^*$ $d(y_n, y_{n+1}) \leq \left(\frac{ka + kb + kc}{1 - kb - kc} \right) d(x_0, x_1)$ dans les cas $x_n = y_n, x_{n+1} = y_{n+1}$; $x_n = y_n, x_{n+1} \neq y_{n+1}$; $x_n \neq y_n, x_{n+1} = y_{n+1}$.

2. Théorème 1. Soient (X, d) un espace métrique complet et métrique convexe, K un sousensemble nonvide, fermé de (X, d) et $T_n : K \rightarrow CB(X)$, $n \in \mathbb{N}^*$, une suite de fonctions multivoques qui satisfont aux conditions (1), (2), (3). Alors la suite (T_n) de fonctions multivoques a un même ensemble de points fixes communs dans K .

Démonstration. 1) Parce que $T_n(x) \subset K$ pour tout $x \in \partial K$, et $n \in \mathbb{N}^*$, il suit qu'il existe un $x_0 \in K$ (ou de ∂K) tel que $T_1(x_0) \subset K$. Si $a = b = c = 0$, $z \in T_1(x_0)$ est un point fixe commun.

En effet, pour $n = 2, 3, \dots$, $H(T_1(x_0), T_n(z)) = 0$ implique $D(z, T_n(z)) = 0$ et donc $z \in T_n(z)$, $n = 2, 3, \dots$

Mais $H(T_2(z), T_1(z)) = 0$ implique $D(z, T_1(z)) = 0$ et alors $z \in T_1(z)$. Ainsi nous pouvons supposer que $a + b + c > 0$.

2) Nous montrerons que la suite (x_n) construite dans la Proposition 1 est une suite de Cauchy.

$$d(x_m, x_n) \leq d(x_m, y_{m+1}) + d(y_{m+1}, y_{n+2}) + \dots + d(y_{n-1}, x_n).$$

Alors, de la Conséquence 1, nous obtenons

$$d(x_m, x_n) \leq \left(\frac{ka + kb + kc}{1 - kb - kc} \right)^{m-1} d(x_0, x_1) + \left(\frac{ka + kb + kc}{1 - kb - kc} \right)^m d(x_0, x_1) + \dots + \left(\frac{ka + kb + kc}{1 - kb - kc} \right)^{n-1} d(x_0, x_1) = \left[\left(\frac{ka + kb + kc}{1 - kb - kc} \right)^{m-1} + \dots + \left(\frac{ka + kb + kc}{1 - kb - kc} \right)^{n-1} \right] d(x_0, x_1).$$

La somme d'entre parenthèses représente le reste partiel d'une progression géométrique avec la ration $\frac{ka + kb + kc}{1 - kb - kc} < 1$ (conformément à l'hypothèse (1)), donc $d(x_m, x_n) \rightarrow 0$ pour $m, n \rightarrow \infty$ et ainsi la suite (x_n) est une suite de Cauchy.

3) La suite (x_n) est convergente à un point $z \in K$. L'espace (X, d) étant un espace métrique complet, la suite (x_n) est convergente à un point $z \in X$ et, K étant un ensemble fermé et tous les $x_n \in K$, il suit que $z \in K$.

4) Nous montrerons maintenant que z est un point fixe pour toutes les fonctions multivoques (T_n) .

De la définition de la suite (x_n) il suit qu'il existe une sous-suite (x_{q_n}) de (x_n) telle que $x_{q_n} = y_{q_n}$ ($n=1, 2, \dots$).

Alors pour $n=1, 2, 3, \dots$ on a :

$$\begin{aligned} D(x_{q_k}, T_n(z)) &\leq H(T_{q_k}(x_{q_{k-1}}), T_n(z)) \leq kad(x_{q_{k-1}}, z) + \\ &+ kb\{D(x_{q_{k-1}}, T_{q_{k-1}}(x_{q_{k-1}})) + D(z, T_n(z))\} + kc\{D(x_{q_{k-1}}, T_n(z)) + D(z, T_{q_k}(x_{q_{k-1}}))\} \\ &\leq ak\{d(x_{q_{k-1}}, x_{q_k}) + d(x_{q_k}, z)\} + kb\{d(x_{q_{k-1}}, x_{q_k}) + d(z, x_{q_k}) + D(x_{q_k}, T_n(z))\} + \\ &+ kc\{d(x_{q_{k-1}}, x_{q_k}) + D(x_{q_k}, T_n(z)) + d(z, x_{q_k})\}. \end{aligned}$$

Ainsi

$$D(x_{q_k}, T_n(z)) \leq \frac{ka + kb + kc}{1 - kb - kc} \{d(x_{q_{k-1}}, x_{q_k}) + d(z, x_{q_k})\},$$

et donc $D(x_{q_k}, T_n(z)) \rightarrow 0$ pour $q_k \rightarrow \infty$.

L'inégalité $D(z, T_n(z)) \leq d(x_{q_k}, z) + D(x_{q_k}, T_n(z))$ et les résultats antérieurs impliquent que $D(z, T_n(z)) = 0$. Mais $T_n(z)$ est un ensemble fermé et alors $z \in T_n(z)$, $n=1, 2, \dots$ et donc z est un point fixe commun pour T_n , $n=1, 2, \dots$

5) Soient $F_k(T_i) = \{x \mid x \in K, x \in T_i(x)\}$, $i \in \mathbb{N}^*$. Nous allons montrer que $F_K(T_i) = F_K(T_j)$, $i, j \in \mathbb{N}^*$ de la même manière que V. Popa [3].

Soit $u \in F_K(T_i)$; alors $u \in T_i(u)$ et, si $D(u, T_j(u)) \neq \emptyset$, alors par (2) nous avons :

$$\begin{aligned} D(u, T_j(u)) &\leq H(T_i(u), T_j(u)) \leq ad(u, u) + b\{D(u, T_i(u)) + D(u, T_j(u))\} + \\ &+ c\{D(u, T_j(u)) + D(u, T_i(u))\}. \end{aligned}$$

Il suit que $D(u, T_j(u)) \leq (b+c)D(u, T_j(u))$, mais $a+2b+2c < 1$ et donc $b+c < (1-a)/2 < 1$. Alors $D(u, T_j(u)) < D(u, T_j(u))$, ce qui est une contradiction, donc $D(u, T_j(u)) = \emptyset$, c'est à dire $u \in T_j(u)$ parce que $T_j(u)$ est fermé.

Nous avons démontré, dans les conditions données, que $F_K(T_i) \subset F_K(T_j)$. Par un raisonnement analogue, il suit que $F_K(T_j) \subset F_K(T_i)$ et donc $F_K(T_i) = F_K(T_j)$.

3. Observations et exemples. 1. On peut obtenir des résultats analogues au Théorème 1 en considérant les fonctions multivoques $T_n: K \rightarrow \text{Cpt}(X)$ ($\text{Cpt}(X)$ est l'ensemble des sousensembles nonvides, compacts de X) pour une seule fonction multivoque $T: K \rightarrow CB(X)(\text{Cpt}(X))$, pour une paire de fonctions multivoques $T_1, T_2: K \rightarrow CB(X)(\text{Cpt}(X))$, ou pour un nombre fini de fonctions multivoques $T_1, T_2, \dots, T_p: K \rightarrow CB(X)(\text{Cpt}(X))$.

Dans les cas $T_n: K \rightarrow \text{Cpt}(X)$, dans le Lemme 2 on doit considérer $k=1$.

2. Aussi nous pouvons énoncer comme conséquences du Théorème 1 des théorèmes de point fixe commun pour une suite de fonctions $T_n : K \rightarrow X$, $n \in \mathbb{N}^*$.

Dans le cas $T_n = T : K \rightarrow X$, du Théorème 1 nous obtenons le suivant

Théorème 2. Soient (X, d) un espace métrique complet et métrique convexe, K un sousensemble nonvide, fermé de X et la fonction $T : K \rightarrow X$ qui satisfait aux conditions (1), (2), (3). Alors la fonction T a un point fixe unique $z \in K$.

Nous donnons maintenant les exemples suivants pour illustrer le Théorème 2.

Exemple 1. Soient $X = [0, 3]$ avec la distance $d(x, y) = |y - x|$, $\forall x, y \in X$ et $K = [\frac{1}{2}, 2]$, ($K \subset X$ et K est un sousensemble fermé de X).

Nous définissons la fonction $T : K \rightarrow X$ de la manière suivante : $T(\frac{1}{2}) = 0$; $T(x) = 1, \frac{1}{2} < x < \frac{3}{2}$; $T(\frac{3}{2}) = 3$ et $T(x) = 1, \frac{3}{2} < x \leq 2$. On voit que $\partial K = \{\frac{1}{2}, 2\}$ et $T(\partial K) = \{1\} \subset K$.

Ils existent aussi les nombres $a = \frac{1}{4}$, $b = c = \frac{1}{8}$ tels que les conditions (1) ... (3) du Théorème 3 soient satisfaites.

Alors la fonction T considérée a le point fixe unique $x = 1 \in [\frac{1}{2}, 2]$.

Exemple 2. Soit l'espace $X = \{a, b\}$ avec la métrique discrète, c'est à dire $d(a, a) = d(b, b) = 0$ et $d(a, b) = 1$. Soit $K = \{b\}$ un sousensemble fermé de X . On définit la fonction $T : K \rightarrow X$ par $T(b) = a$.

Alors, car T ne satisfait pas à toutes les conditions du Théorème 2 ($\partial K = K$ et $T(\partial K) \not\subset K$), T n'a pas de point fixe.

3. On peut obtenir aussi des résultats analogues aux Théorèmes 1 et 2 de cette Note dans le cas $K = X$.

4. Si (X, d) est un espace de Banach, alors des Théorèmes 1 et 2 nous obtenons d'autres théorèmes analogues de points fixes communs non-unique ou unique (pour une suite de fonctions $T_n : K \rightarrow X$, $K \subset X$, ou d'opérateurs, quand $K = X$, ou pour une seule fonction).

5. La méthode employée par nous dans cette Note est différente de celle de N. Assad [1] car elle est une méthode constructive qui nous permet de montrer directement que (x_n) est une suite de Cauchy.

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BIBLIOGRAPHIE

1. Assad N., *A Fixed Point Theorem for Weakly Uniformly Strict Contractions*. Canad. Math. Bull., 16 (1) (1973).
2. Itoh S., *Multivalued Generalized Contractions and Fixed Point Theorems*. Research Reports on Information Sciences, Tokyo Institute of Technology, A-21 (1975).
3. Popa V., *Common Fixed Points for Multifunctions Satisfying a Rational Inequality*. Kobe J. Math., 2, 23-28 (1985).
4. Yanagi K., *A Common Fixed Point Theorem for a Sequence of Multivalued Mapping*. Publ. RIMS Kyoto University, 15, 47-52 (1979).

ASUPRA UNOR TEOREME DE PUNCTE FIXE COMUNE PENTRU MULTIFUNCȚII

(Rezumat)

Se rezolvă o problemă deschisă, propusă de K. Yanagi, înlocuind condiția $a + (a + 3)(b + c) < 1$, $a, b, c \geq 0$, prin condiția mai relaxată $a + 2b + 2c < 1$, $a, b, c \geq 0$, în teoremele de puncte fixe comune ale lui S. Itoh [2] și K. Yanagi [4].

SYSTEMS OF EVOLUTION INEQUALITIES WITH IRREGULAR IMPLICIT OBSTACLE

BY

ROBERT T. VESCAN

1. In what follows, we construct solutions which we call „almost strong“ for parabolic variational inequalities with essentially *irregular* implicit obstacles. Such solutions are intermediate between the known „weak“ and „strong“ solutions. The existence of strong solutions has been proved when the obstacle operator satisfies some regularity properties: order monotonicity in the particular case of stochastic impulse control problems, see e.g. [1], [3], [6], or a kind of norm boundedness and possibility to resort to Lewy-Stampacchia dual estimates in [4]. We are concerned with the situation when such hypotheses are not valid and the obstacles are *only a.e. defined, not even in $L^2(Q)$* . We prove the existence of „approximate“ solutions u_n and we infer that a subsequence of u_n converges weakly to an „almost“ strong solution u (see the definition below). This result is applicable in problems with differential operators and eventually to implicit complementarity problems [5], [7].

2. For a bounded open regular cylinder $Q = (0, T) \times D$ in $\mathbb{R}^{N+1}$, we consider the reflexive Banach space $L^p = L^p(0, T; W^{1,p}(D))$, $p \geq 2$. Let us denote the norm in L^p and its dual $L^q = L^q(0, T; W^{1,q}(D)')$, $q = p/(p-1)$ with $\|\cdot\|$. We set $W(0, T) = \left\{ v \in L^p; \frac{dv}{dt} \in L^q \right\}$ with the derivative $\frac{dv}{dt}$ in the sense of distributions; $W(0, T)$ is known to be a reflexive Banach space with the norm $\|v\|_W = \|v\| + \left\| \frac{dv}{dt} \right\|$.

Definition. Let A be a mapping from L^p to L^q , $M: L^p \rightarrow F(Q, \mathbb{R})$ an obstacle operator with values in the spaces of classes of real functions defined Lebesgue a.e. Suppose that f belongs to L^p . We call $u \in L^p$ an *almost strong solution* for the system of parabolic inequalities associated to (A, M, f) if there exist $\tilde{u}$ in the dual space $W(0, T)'$ and $v \in L^1(D)$ such that

$$(1) \quad \begin{cases} u \in L^p, u \leq Mu \text{ a.e. in } Q; \\ \tilde{u}(w) = \int_D v w(T) dx - \left\langle \frac{dw}{dt}, u \right\rangle, \quad \forall w \in W(0, T); \\ \langle Au - f, w - u \rangle + \tilde{u}(w) - \frac{1}{2} \int_D v^2 dx \geq 0 \text{ for any } w \in W(0, T), \\ w \leq Mu \text{ with } w(0) = 0. \end{cases}$$

Proposition 1. *If u is an almost strong solution then it verifies the system of weak inequalities*

$$(2) \quad \langle Au + \frac{dw}{dt} - f, w - u \rangle - \frac{1}{2} \|v - w(T)\|_{L^1(D)}^2 \geq 0$$

for any $w \in W(0, T)$, $w \leq Mu$, but only those with $w(0) = 0$.

If moreover u is regular, i.e. $u \in W(0, T)$, $\tilde{u} \in L^q$, then it verifies the strong inequalities

$$(3) \quad \langle Au + \frac{du}{dt} - f, w - u \rangle \geq 0 \text{ for } w \in W(0, T), w \leq Mu, w(0) = 0$$

and $u(T) = v, u(0) = 0$.

Proof. (1) gives for any $w \in W(0, T)$, $w(0) = 0$, $w \leq Mu$ the inequalities

$$\langle Au - f, w - u \rangle + \int_D v w(T) dx - \left\langle \frac{dw}{dt}, u \right\rangle - \frac{1}{2} \|v\|_{L^1(D)}^2 \geq 0$$

and further, using integration by parts,

$$\langle Au + \frac{dw}{dt} - f, w - u \rangle + \int_D v w(T) dx - \frac{1}{2} \|v\|_{L^1(D)}^2 - \frac{1}{2} \|w(T)\|_{L^1(D)}^2 \geq 0.$$

These inequalities imply the weak relations (2).

If u and $\tilde{u}$ are regular, one takes $w \in W(0, T)$ with $w(0) = w(T) = 0$ and obtains $\tilde{u}(w) = -\langle dw/dt, u \rangle = \langle du/dt, w \rangle$; therefore we may identify $\frac{du}{dt} = \tilde{u}$ in $L^q \subset W(0, T)'$. Thus, for $w \in W(0, T)$, $w(0) = 0$, we may write

$$\tilde{u}(w) = \left\langle \frac{du}{dt}, w \right\rangle = \int_D v w(T) dx - \left\langle \frac{dw}{dt}, u \right\rangle$$

which yields $\int_D u(T) w(T) dx = \int_D v w(T) dx$.

We conclude that $u(T) = v$ in $L^2(D)$. Similarly, for $w \in W(0, T)$, we

get $\int_D u(0)w(0)dx=0$, which shows that $u(0)=0$ in $L^2(D)$. Next, from (1) it follows

$$\langle Au-f, w-u \rangle + \langle \frac{du}{dt}, w \rangle - \langle \frac{du}{dt}, u \rangle \geq 0 \text{ for suitable}$$

$w \in W(0, T)$, $w(0)=0$, $w \leq Mu$.

Remark 1. Conversely, if u verifies the usual strong inequalities (3), then u is an almost strong solution (1) with $\tilde{u}=du/dt$ and $v=u(T)$.

Proposition 2. Let us suppose that:

(A1) A is coercive: $\exists c_1 > 0$, $\langle Av, v \rangle \geq c_1 \|v\|^p$, $\forall v \in L^p$;

(A2) A is bounded: $\exists c_2 > 0$, $\|Av\| \leq c_2 \|v\|^{p-1}$, $\forall v \in L^p$;

(A3) A is monotone hemicontinuous;

(M1) $Mu \geq 0$ for any $u \in L^p$ with $\|u\| \leq r = (\|f\|_{L^q}/c_1)^{1/p-1}$;

(M2) M is sequentially continuous: if $v_i \rightarrow v$ in the norm of $L^2(Q)$, then a subsequence of Mv_i converges a.e. on Q to Mv ;

(M3) if $y_i \rightarrow u$ weakly in L^p , then each $w \in W(0, T)$, $w \leq Mu$, $w(0)=0$ is the limit of a $W(0, T)$ -strongly convergent subsequence $w_i \leq My_i$, $w_i(0)=0$. Then for $\alpha_n \nearrow \infty$, there is a sequence of approximate solutions (u_n) :

$$(4) \quad \begin{cases} u_n \in W(0, T), u_n(0)=0, u_n \leq Mu_n \text{ a.e. in } Q, \\ \langle Au_n + \frac{du_n}{dt} - f, w - u_n \rangle \geq 0, \forall w \in W(0, T), \left\| \frac{dw}{dt} \right\| \leq \alpha_n, \\ w(0)=0, w \leq Mu_n. \end{cases}$$

Proof. Let us consider the selection map

$$S: L^p \rightarrow K = \left\{ v \in W(0, T) ; \left\| \frac{dv}{dt} \right\| \leq \alpha_n, v(0)=0 \right\} \text{ defined by}$$

$v \in S(u)$ if $v \leq Mu$, $v \in K$, $\langle Av + dv/dt - f, w - v \rangle \geq 0$, $\forall w \in K$, $w \leq Mu$.

The set $C(u) = \{v \in K ; v \leq Mu\}$ is convex closed in L^p since if $v_i \in C(u)$, $v_i \rightarrow v$ in L^p then, by the fact that $\frac{d}{dt}: \{v \in W(0, T) ; v(0)=0\} \rightarrow L^q$ has

(convex) closed graph, it follows that $dv_i/dt \rightarrow dv/dt$ weakly in L^q and; $v(0)=0$; furthermore, one has a subsequence $v_i \rightarrow v$ a.e. on Q , hence $v_i \leq Mu$ yield $v \leq Mu$. The operator $A + d/dt$ is coercive for $v \in C(u)$: $\langle Av + dv/dt, v \rangle = \langle Av, v \rangle + \frac{1}{2} \|v(T)\|_{L^2(D)}^2$. Thus the existence of $v \in S(u)$ follows by Browder-Hartman-Stampacchia's theorem [2].

The statement of the proposition is just that S has a fixed point. Let us now observe that by a known Minty's Lemma [2], $S(u)$ is convex closed in L^p . Recall that by (M1) $0 \in C(u)$, $\forall u \in L^p$, $\|u\| \leq r$; as it is easy to see $\langle Av + dv/dt - f, 0 - v \rangle < 0$ if $v \in K$, $\|v\| > r$; thus we have $v \in S(u) \subset K \cap B_r$, $\forall u \in B_r$, where B_r is the ball centered at 0 in L^p , with radius r . Obviously, $K \cap B_r$ is weakly compact in L^p . Let us take $y_i \rightarrow u$ and $v_i \in S(y_i)$, $v_i \rightarrow v$ both in $K \cap B_r$. The embedding $L^p \subset L^2(Q)$ is compact, therefore $v_i \leq My_i$ and (M2) imply $v \leq Mu$. For $w \in C(u)$, there exists by

(M3) a subsequence $w_i \in C(y_i)$, $w_i \rightarrow w$ in $W(0, T)$. Since $v_i \in S(y_i)$, we have $\langle Av_i + dv_i/dt - f, w_i - v_i \rangle \geq 0$; then

$$\overline{\lim} \langle Av_i + \frac{dv_i}{dt}, w - v_i \rangle + \overline{\lim} \langle Av_i + \frac{dv_i}{dt}, w_i - w \rangle \geq \langle f, w - v \rangle.$$

$(Av_i + dv_i/dt)$ is bounded and the first $\overline{\lim}$ is 0; by monotonicity, it follows $\langle Aw + dw/dt, w - v \rangle \geq \langle f, w - v \rangle$, $\forall w \in C(u)$ and by the mentioned Lemma this means $v \in S(u)$. We conclude that S has weak closed graph in $(K \cap B_r) \times (K \cap B_r)$ and F a n's fixed point theorem works in this situation.

Remark 2. An obstacle operator of the form

$$(Mv)(t, x) = h_1(t, x) + h_2(t, x) \cdot h_3(v),$$

with $h_1 \in F(Q, \mathbb{R})$, $h_2 \in W(0, T)$, $h_2(0) = 0 \in L^2(D)$ and h_3 continuous from $L^2(Q)$ -topology into $\mathbb{R}$, may verify (M1), (M2), (M3). E.g. for (M3) one may take

$$w_i = w + My_i - Mu \text{ for each } w \in W(0, T), w(0) = 0, w \leq Mu.$$

Theorem. Under the hypotheses (A1) (A2) (A3) (M1) (M2) (M3) and (A4) for a sequence $u_i \rightarrow u$ weakly in L^p , one has

$$\overline{\lim} \langle Au_i, w - u_i \rangle \leq \langle Au, w - u \rangle, \quad \forall w \in L^p,$$

then the sequence of approximate solutions u_n (4) constructed in Proposition 2 has a weak cluster point u which is an almost strong solution in the sense (1).

Proof. By the argument from the previous proof, one finds that $u_n \in S(u_n) \subset B_r$. Besides, the inequalities (4) for $w = 0$ give $\langle Au_n - f, -u_n \rangle \geq \langle du_n/dt, u_n \rangle \geq 0$; these inequalities, together with (A2) imply that $\langle du_n/dt, u_n \rangle = \frac{1}{2} \|u_n(T)\|_{L^2(D)}^2$ is bounded by a constant dependent on c_1 , c_2 , $\|f\|_{L^q}$. Because of

$$\left| \left\langle \frac{du_n}{dt}, w \right\rangle \right| \leq \|u_n(T)\|_{L^2(D)} \cdot \|w(T)\|_{L^2(D)} + \left\| \frac{dw}{dt} \right\| \cdot \|u_n\|, \quad \forall w \in W(0, T)$$

it follows that (du_n/dt) is bounded in the dual $W(0, T)'$. Therefore a subsequence $u_n \rightarrow u$ weakly in L^p , $du_n/dt \rightarrow \tilde{u}$ weakly in $W(0, T)$, $u_n(T) \rightarrow v$ weakly in $L^2(D)$. The equalities

$$\tilde{u}(w) = \int_D v w(T) dx - \left\langle \frac{dw}{dt}, u \right\rangle$$

$$\text{can be deduced from } \left\langle \frac{du_n}{dt}, w \right\rangle = \int_D u_n(T) w(T) dx - \left\langle \frac{dw}{dt}, u_n \right\rangle, \quad \forall w \in$$

$\in W(0, T)$. It remains to show that $\langle Au - f, w - u \rangle + \tilde{u}(w) - \frac{1}{2} \|v\|_{L^2(D)}^2 \geq 0$ for $\forall w \in W(0, T)$, $w \leq Mu$, $w(0) = 0$; by (M3) there is a strongly convergent subsequence $w_{n_i} \rightarrow w$ in $W(0, T)$ with $w_{n_i} \leq Mu_{n_i}$, $w_{n_i}(0) = 0$. Since $\left\| \frac{dw_{n_i}}{dt} \right\| \rightarrow$

$\rightarrow \left\| \frac{dw}{dt} \right\|, \left\| \frac{dw_{n_i}}{dt} \right\| \leq \alpha_{n_i}$, for large n_i . Then, according to (4) one gets

$$\langle Au_{n_i} + \frac{du_{n_i}}{dt} - f, w_{n_i} - u_{n_i} \rangle \geq 0.$$

Because $\lim \|u_{n_i}(T)\|_{L^2(D)}^2 \geq \|v\|_{L^2(D)}^2$, taking the $\overline{\lim}$ in these relations, we obtain

$$\begin{aligned} \overline{\lim} \langle Au_{n_i}, w - u_{n_i} \rangle + \overline{\lim} [\langle Au_{n_i}, w_{n_i} - w \rangle + \langle \frac{du_{n_i}}{dt}, w_{n_i} \rangle - \\ - \frac{1}{2} \|u_{n_i}(T)\|^2 - \langle f, w_{n_i} - u_{n_i} \rangle] \geq 0. \end{aligned}$$

Thus, it remains to apply the assumption (A4).

Remark 3. The weak u.s.c. condition (A4) holds, for instance, if the operator A is either completely continuous or linear (and bounded-A2, positive-A1).

Remark 5. In case that both u and Mu are regular [8] ($Mu \in L^p$, $d(Mu)/dt \in L^2(Q)$) then $u(1)$ verifies the strong system $\langle Au + du/dt - f, w - u \rangle \geq 0$, for all $w \in L^p$, $w \leq Mu$.

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REFERENCES

1. Bensoussan A., Lions J. L., *Contrôle impulsionnel et inéquations quasi variationnelles*, Ch. IV. Dunod-Paris, 1982.
2. Browder F. E., *Nonlinear Monotone Operators and Convex Sets in Banach Spaces*. Bull. Amer. Math. Soc., 71, 780–785 (1965).
3. Charrier P., *Contribution à l'étude de problèmes d'évolution*. Thèse, Univ. Bordeaux I, 1978.
4. Donati F., Matzeu M., *On Some Evolution Problems of Implicit Type*. Boll. Un. Mat. Ital., 5, 17-B, 478–491 (1980).
5. Isac G., *Problèmes de complémentarité*. Univ. Limoges, 1985.
6. Matzeu M., Vivaldi M. A., *On the Regular Solution of a Nonlinear Parabolic Quasi-variational Inequality Related to a Stochastic Control Problem*. Comm. Partial Diff. Eq., 4 (10), 1123–1147 (1979).
7. Vescan R. T., *Un problème variationnel implicite faible*. C. R. Acad. Sc. Paris, 299 Sér. I, 655–658 (1984).
8. Vescan R. T., *Parabolic Implicit Problems with Irregular Constraints*. Proc. Conf. Diff. Equations, Cluj-Napoca, Nov. 1985.

SISTEME DE INEGALITĂȚI DE EVOLUȚIE CU OBSTACOL IMPLICIT NEREGULAT

(Rezumat)

Se construiesc soluții „aproape tari” pentru inegalități variaționale parabolice cu obstacole implicite neregulate mărginite în sensul ordinii.

SHORT SURVEY ON AN AUTOMATIC SOLVING OF GEOMETRICAL OPTIMIZATION PROBLEMS WITH PARTICULAR STRUCTURE

BY

D. LEPĂDATU

1. General Remarks. Any attempt at the elaboration of certain computer-assisted flight control systems, calls for the existence of a representation of the resolution processes so that these ones or at least parts of them might be simulated. Research work in this field has always aimed at both the grasping of the phenomena and the use of a mathematical structure that should thus thoroughly describe them. Our governing idea is that any complex problem is a system by itself which must be put under man's decident control and that on the data base of the controlled system there are modules for solving certain particular classes of problems. Looked upon from the vista of the two complexity functions: the time function and the space function, the above remarks lead to the simplification of the global solving of the problem and makes efficient the mapping of both the mathematical programming and the computer programming in the study of phenomena.

Most of the geometrical optimization problems with particular form have been solved by means of general algorithms, a fact which made them appear practically unsolved for control systems with a short time-answer.

Nowadays, any computer has in its reference library at least one package assigned to the linear programming it usually being the most performant piece in the computer's software range of applications.

Applying the hierarchy factor in the field of modelling, the complex model will be replaced by a set of more than one moduli of varying shapes, placed at different levels of the decomposition aggregation process, each modulus thus becoming an independent unit constituted with well defined functions, strictly connected with the other moduli.

Theoretically, the solution thus found is inferior to the solution furnished by the global model but it has the advantage of being closer to reality if we take into account the application conditions or the answer times.

Machine tools design, the establishment of optimal machine tools work rates, the solving of optimization multicriterial problems on the basis of certain supplementary criteria, the examination of production systems in which both the "output" function and the restrictions are pasinomes, models of production increment in agriculture by means of natural and artificial multiple factors, production quality control models of automatic systems, are but a few of the domains in which geometrical optimization proves to be a powerful tool [1], [2], [10], [11].

In what follows we present an automatic procedure for solving a certain class of geometrical optimisation problems with particular structure.

2. Presentation of the Problem and its Transformation. In the course of an activity of any kind, on the basis of the theoretical-experimental results, there have been determined the work parameters, the variable magnitudes and their limits. Dependences and optimal chosen criteria have a pozinome form. Below we render the general model of the problems which will be solved (optimal)

$$(1) \quad X_1^{\alpha_{01}} X_2^{\alpha_{02}} \dots X_n^{\alpha_{0n}}$$

where the variables $X_1, X_2, \dots, X_n$ satisfy the restrictions system (2) + (3)

$$X_1^{\alpha_{11}} X_2^{\alpha_{12}} \dots X_n^{\alpha_{1n}} \leq c_1,$$

$$(2) \quad X_1^{\alpha_{21}} X_2^{\alpha_{22}} \dots X_n^{\alpha_{2n}} \leq c_2,$$

$$\dots \dots \dots$$

$$X_1^{\alpha_{m1}} X_2^{\alpha_{m2}} \dots X_n^{\alpha_{mn}} \leq c_m,$$

with

$$(3) \quad c_i > 1, \quad i=1, 2, \dots, m,$$

$$1 \leq a_j \leq X_j \leq b_j,$$

for $j=1, 2, \dots, n$ and X_j variables not all of them, by all means upper limited and α real numbers.

A general presentation of geometrical optimization problems with solving algorithms are delineated in [1].

Working out the transformation $X_j = e^{Z_j}$, $j=1, 2, \dots, n$, problem (1), (2) and (3) becomes

$$(1') \quad (\text{optimal}) \quad \alpha_{01} Z_1 + \alpha_{02} Z_2 + \dots + \alpha_{0n} Z_n,$$

$$\alpha_{i1} Z_1 + \alpha_{i2} Z_2 + \dots + \alpha_{in} Z_n \leq c_i,$$

with

$$(2') \quad c_i = \ln c_i, \quad i=1, 2, \dots, m,$$

$$(3') \quad 0 \leq a'_j \leq Z_j \leq b'_j, \quad a'_j = \ln a_j, \quad b'_j = \ln b_j$$

for $j=1, 2, \dots, n$.

One may notice that the problem (1')—(3') is a linear programming problem for which we have powerful solving algorithms. If $Z_1, Z_2, \dots, Z_n$ is the solution of the problem (1')—(3'), then $X_1 = e^{Z_1}, \dots, X_n = e^{Z_n}$, is the solution to the problem (1)—(3).

3. The Processing Algorithm. The object under study is considered to be depicted in the problem (1)—(3) and we have a satisfactory number information in the data base of the system in which we want to integrate it. More precisely, if the whole thing refers to any activity belonging to the programming of production itself, then we possess enough information in

the data base of the respective computerized system. In case it refers to special device projection, these devices there are well enough described in a technical data base.

We also notice that a processing algorithm capable to give a solution to a problem is more or less efficient strictly depending on the way the structures of data, the solving algorithm, and the control sequences are or aren't the most indicated. In our case, the automatic procedure must solve the problem for big m , n dimensions, that is, there were general algorithms grow practically inapplicable.

For the representation of the problem we write below the definition of a linear structure, using COBOL language particular notations :

01	ARTP	03	CODV
02	OPTIM	03	VALI
03	CODF	03	VALS
03	MAXI	03	VALO
03	VALF	02	LEG OCCURS $n \cdot m + n$
02	CODR OCCURS m	03	CODVL
03	REST	03	RESTV
03	CEK	03	EXPO
02	VARB OCCURS n		

where : CODF — is the code of the functions to be optimized ; MAXI — contains information about maximal or minimal ; VALF — the optimized function value ; REST — the restriction code ; CEK — C_i quantity from problem (1)–(3) ; CODV — the variable code ; VALI — a_j inferior value ; VALS — b_j superior value ; VALO — the optimal value ; CODVL the j variable code ; RESTV — the restriction K in which appears the j variable, the criterium function inclusively ; EXPO — α_{kj} exponent.

It may be noticed that level 02 OPTIM carries information about the optimal criterium ; CODR — contains information about restrictions ; VARB — contains information about variables ; LEG — contains α_{kj} matrix with $k=0, 1, 2 \dots m, j=1, 2, \dots, n$.

It is obvious that the ARTP level 01 structure contains the problem (1)–(3) and the above representation makes possible the OPALINE programme application. The work instructions are supposed to be known.

The procedural description is assumed in order to provide the work steps leading to formalization, therefore the algorithm [5], [6] synthesis.

Step 1. Out of the data base extract the α_{kj} , a_j , b_j , c_j coefficients, logically validate them and fill in the ARTP structure. Work on a data base calls for the OPALINE input file construction straightly from ARTP. Independent work requires the use of files so as to ensure the generality of the procedure. In what follows, the work will be performed by means of a file in which each issue of new datum at 02 level represents its record. This file is called PROBL.

Step 2. Read PROBL file, at end go to step 4. If the record is one of OPTIM type : Write ROWS record ; Write N and CODF record ; retain MAXI ; go to step 2. If it is a CODR type record, then write record with L and REST ; go to step 2. If records of this type are over, write RHS record ; fix the file at first CODR record.

Step 3. Read PROBL file, at end go to step 4. If it is a CODR type record, write a record with information from REST and In (CEC); go to step 3. If it is a LEG type record (at its first issue write COLUMNS record), write current record with information taken from CODL, RESTV, EXPO; go to step 3. If it is a VARB type record (at its first issue write BUNDS-record, write a current record with LO CODV, In (VALI) information, if the variable has inferior limit and another current record with "UP", CODV, In (VALS) if the variable has superior limit; go to step 3.

Step 4. Write ENDATA, MAXI (MINI), START records.

Step 5. Run OPALINE programme.

Step 6. Process the OPALINE output file. Fill in VALF and VALO solution values with the solution exponential got from OPALINE. List problem solution.

The file in which we write is undoubtedly OPALINE input. Provided that the PROBL file was independently created, then steps 2, 3, 4 may be grouped together in a programme. Step 5, 6 constitute by themselves, each a separate programme. In this case we can build a simple automatic procedure made up by the respective programmes, linking information being stored in transaction file.

4. One Example. Conclusions. Machine-tool rational exploitation plays a major part in the field of machine making. Thus — sharpening — is one of the most important operations of the splintering tool making operating process. It is on the way this is performed that the tool splintering capacity depends as well as their durability, the splintering rates, the rugosity of the processing surfaces, the productivity and the processing economic efficiency [2]. Recent research in this field makes excessive use of such mathematical models, still, all of them miss an automatic working procedure. Several solving attempts have been made for a small number of variables with nonlinear methods and in a high computer time.

Among the mathematical models established for the improvement of the machine tool work rates, lies the one we present below:

$$X_1^{-0.35} X_2^{0.83} X_3^{0.93} X_4^{0.33} X_5^{0.07} \leq 7.721.88,$$

$$X_1^{0.39} X_2^{0.20} X_3^{-0.09} X_4^{0.13} X_5^{0.12} \leq 13.306,$$

$$X_1^{-0.19} X_2^{0.28} X_3^{-0.10} X_4^{0.10} X_5^{0.10} \leq 2,$$

$$X_1^{-0.41} X_2^{0.94} X_3^{-0.09} X_4^{0.36} X_5^{0.09} \leq 574.19,$$

$$X_1^{0.50} X_2^{-0.23} X_3^{-0.16} X_4^{0.16} X_5^{0.09} \leq 7.710,$$

$$X_1^{0.33} X_2^{0.22} X_3^{-0.27} X_4^{0.20} X_5^{0.13} \leq 4.60,$$

$$1 \leq X_1 \leq 100,$$

$$1 \leq X_2 \leq 24,$$

$$6.76 \leq X_3 \leq 31.20,$$

$$4 \leq X_4 \leq 32,$$

$$3 \leq X_5 \leq 36,$$

$$(\min) X_1^{-0.15} X_2^{-0.15} X_3^{-0.04} X_4^{0.08} X_5^{0.05}$$

Given the new procedure conditions, optimization lasted approximatively 4 minutes, with solution $X_1=66.803$; $X_2=23.900$; $X_3=31.090$; $X_4=4.001$; $X_5=3.020$ and the 0.34 criterion value, a fact that leads to a competitive result the variables significance and the optimal criterion significance being taken into account [2].

Experimental research in system study has demonstrated that a quality and efficiency decision is the result of a several criteria, including the introduction of information processing new technique. Therefore, this procedure can be extended and improved in order to work with any, but finite number of criteria. Extensions can be made to work with a larger number of c free terms vectors, as well as to rise the ≥ 1 variables restrictions. In what follows we stick to a basis of particular or general models which the human factor might conversationally handle in any controlling or designing activity.

Consequently, we express our belief that information science must become more implied, that the results and models shown above must be drawn closer to the users that man should never be eliminated from these systems and replaced by models, since, he is the very decisive factor the flexible element that ensures quality, efficiency and competitiveness to any activity.

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REFERENCES

1. Marușciac I., *Programare geometrică și aplicații*. Ed. „Dacia”, Cluj-Napoca, 1978.
2. Gherghel N., *Contribuții teoretice și experimentale privind optimizarea regimurilor de ascuțire prin abrazare electrochimică a sculelor așchietoare armate cu plăcuțe din carburi metalice*. Teză de doctorat, Inst. polit. Iași, 1979.
3. Felea V., *Curs de date și bănci de date*. Lito, Iași, 1982.
4. ***OPALINE — Manuel d'Utilisation. C.I.I., Paris, 1972.
5. Goad C., *Automatic Construction of Special Purpose Programs*. LNCS — 138, Springer-Verlag, Berlin, 1982.
6. Guiho G., Gresse C., *Programs Synthesis from Incomplete Specifications*. Proc. 5-th Conf. on Automated Deduction, LNCS, Springer-Verlag, Berlin, 1982.
7. Lepădatu D., *Optimizarea planificării producției la I. M. Nicolina*. Com. „Faza inaugurală Cîntarea României”, 25—26 martie 1978.
8. Lepădatu D., *Optimizarea planificării producției la I. M. Nicolina*. Com. Simp. „Experiența utilizării modelelor matematice în sistemele informatice”. Iași, 20—21 aprilie 1978.
9. Lepădatu D., *Structuri de date în sistemele informatice*. Com. Simp. „Opinii privind perfecționarea sistemului informațional în agricultură”, 23—24 martie 1979.
10. Lepădatu D., Irimia M., *Posibilități de fundamentare a unui pachet de programe în elaborarea variantelor de plan*. St. și cerc. economice, 3 (1979).
11. Lepădatu D., *Algoritmi de aproximare a soluțiilor pentru probleme de optimizare liniară multicriteriale*. Com. Simp. „Probleme de planificare”, 14—15 mai 1977.
12. Lepădatu D., *Considerații privind integrarea algoritmului lui Wolfe în sistemele informatice*. Com. ses. științ. „Calitate, eficiență, competitivitate”, Iași, 11—12 noiembrie 1983.

SCURTĂ PRIVIRE ASUPRA REZOLVĂRII AUTOMATE A PROBLEMELOR DE OPTIMIZARE GEOMETRICĂ CU STRUCTURĂ PARTICULARĂ

(Rezumat)

Se prezintă partea conceptuală a unei proceduri automate pentru rezolvarea problemelor de optimizare geometrică în forma particulară (1)—(3). Fără a se intra în detalii sînt dați pașii algoritmului de prelucrare, structurile de date și se fac referiri la domeniile unde s-a folosit procedura.

FLOW IN MATROIDS WITH PARALLEL ELEMENTS

BY

CĂTĂLINA LOVINESCU

Let M be a matroid with parallel elements defined on the set $S = \{e_0, e_1, \dots, e_n\}$. We suppose that there exist elements parallel to e_0 , and denote them by $e_1, e_2, \dots, e_t$. Then $C_1 = \{e_0, e_1\}$, $C_2 = \{e_0, e_2\}, \dots, C_t = \{e_0, e_t\}$ are circuits in M and they are the only circuits which contain e_0 . So $\mathcal{C}_{e_0} = \{C_1, C_2, \dots, C_t\}$.

We call e_0 -flow admissible in matroid M a function $F: S \rightarrow R_+$ satisfying the conditions

$$(1) \quad l(e_i) \leq F(C_i) \leq L(e_i).$$

$$(2) \quad l(e_0) \leq \sum_{i=1}^t F(C_i) \leq L(e_0).$$

where $F(C_i) = F(e_0) + F(e_i)$, $i = \overline{1, t}$ and $l: S \rightarrow R_+$, $L: S \rightarrow R_+$ are given.

Theorem 1. *The necessary and sufficient condition for the existence in matroid M of an e_0 -flow which satisfies the condition*

$$(3) \quad F(e_i) = F(e_0), \quad i = \overline{1, t},$$

is that

$$(4) \quad \max \left(l(e_i), \frac{l(e_0)}{t} \right) \leq \min \left(L(e_i), \frac{L(e_0)}{t} \right)$$

for any circuit $C_i = \{e_0, e_i\} \in \mathcal{C}_{e_0}$.

Proof. Necessity: We suppose that in M there is an e_0 -flow F which satisfies condition (3). By conditions (1), (2), (3) and Theorems 1, 2, 3 of [4], condition (4) results.

Sufficiency: We suppose condition (4) verified and define

$$F(e_i) = F(e_0) = \frac{1}{2} \max \left(l(e_i), \frac{l(e_0)}{t} \right).$$

So

$$F(C_i) = \max \left(l(e_i), \frac{l(e_0)}{t} \right).$$

Then

$$l(e_i) \leq \max \left(l(e_i), \frac{l(e_0)}{t} \right) = F(C_i) \leq \min \left(L(e_i), \frac{L(e_0)}{t} \right) \leq L(e_i),$$

and

$$l(e_0) \leq t \max \left(l(e_i), \frac{l(e_0)}{t} \right) = t F(C_i) \leq t \min \left(L(e_i), \frac{L(e_0)}{t} \right) \leq L(e_0).$$

which means that F is an e_0 -flow in M .

Theorem 2. *The necessary and sufficient condition for the existence in matroid M of an e_0 -flow with equal values along all circuits of $\mathcal{C}_{e_0}$ is that*

$$(5) \quad \max_{i=1,t} \left(l(e_i) + \frac{l(e_0)}{t} \right) \leq \min_{i=1,t} \left(L(e_i) + \frac{L(e_0)}{t} \right).$$

Proof. *Necessity:* We suppose that in matroid M with parallel elements there is an e_0 -flow with equal values along all circuits of $\mathcal{C}_{e_0}$. Then by Theorem 2 of [4] it results that

$$\max_{i=1,t} \left(l(e_i), \frac{l(e_0)}{t} \right) \leq \min_{i=1,t} \left(L(e_i), \frac{L(e_0)}{t} \right).$$

Hence, it follows that

$$l(e_i) + \frac{l(e_0)}{t} \leq 2 \max_{i=1,t} \left(l(e_i), \frac{l(e_0)}{t} \right) \leq 2 \min_{i=1,t} \left(L(e_i), \frac{L(e_0)}{t} \right) \leq L(e_i) + \frac{L(e_0)}{t}.$$

so condition (5) is verified.

Sufficiency: We suppose condition (5) be satisfied. Let $u \in R_+$ be so that

$$\max_{i=1,t} \left(l(e_i) + \frac{l(e_0)}{t} \right) \leq u \leq \min_{i=1,t} \left(L(e_i) + \frac{L(e_0)}{t} \right).$$

$$u = (t+1) u_0,$$

$$\max_{i=1,t} (l(e_i)) \leq u_0 \leq \min_{i=1,t} (L(e_i)), \quad \frac{l(e_0)}{t} \leq u_0 \leq \frac{L(e_0)}{t}.$$

We define the function $F: \mathcal{C}_{e_0} \rightarrow R_+$ by $F(C_i) = u_0$ and show that it is an e_0 -flow in M .

Indeed

$$l(e_i) \leq \max_{i=1,t} (l(e_i)) \leq u_0 = F(C_i) \leq \min_{i=1,t} (L(e_i)) \leq L(e_i),$$

and

$$l(e_0) \leq t u_0 = t F(C_i) \leq L(e_0).$$

Corollary 1. *The necessary and sufficient condition for the existence in a matroid with parallel elements of an e_0 -flow with equal values along all circuits of $\mathcal{C}_{e_0}$ is that*

$$(6) \quad \max(l(e_i) + l(e_0)) \leq \min(L(e_i) + L(e_0)).$$

Proof. Conditions (5) and (6) are equivalent.

Corollary 2. *From any simple matroid M in which there is an e_0 -flow with equal values along all circuits we can obtain matroids with parallel elements in which there is an e_0 -flow.*

Proof. Let M be a simple matroid on $S = \{e_0, e_1, \dots, e_n\}$ and $\mathcal{C}_{e_0} = \{C_1, C_2, \dots, C_p\}$ the set of its circuits that contain e_0 . We replace any element of circuits $C_j \in \mathcal{C}_{e_0}$ by parallel elements. Thus we obtain a matroid M' defined on the set S' , made up of the elements of S and the added elements. $C_j \in \mathcal{C}_{e_0}$ are dependent sets in this matroid, but they are no longer circuits. If $C_k \notin \mathcal{C}_{e_0}$ and $C_k \cap C_j = \emptyset$, for any $C_j \in \mathcal{C}_{e_0}$, then C_k remains a circuit in the new matroid M' . If $C_k \notin \mathcal{C}_{e_0}$ and there is $C_j \in \mathcal{C}_{e_0}$ so that $C_k \cap C_j \neq \emptyset$ then C_k is a dependent set in M' but it is no longer a circuit. The set made up of two parallel elements of S' are circuits in M' . It can be easily verified, on the basis of axioms (C_1) , (C_2) of [5], that M' defined on S' , with the set of circuits described above, is a matroid.

In M' the set of the circuits containing e_0 is $\mathcal{C}'_0 = \{C'_1, C'_2, \dots, C'_t\}$, where $C'_1 = \{e_0, e'_1\}$, $C'_2 = \{e_0, e'_2\}$, ..., $C'_t = \{e_0, e'_t\}$, $e'_1, e'_2, \dots, e'_t$ being all the elements of S' parallel to e_0 . If in the simple matroid M there is an e_0 -flow, then on the basis of Theorems 1, 2, 3 of [4] and of Theorem 1 of this paper, it results that there is an e_0 -flow in matroid M' with parallel elements.

Let M' be a matroid with parallel elements defined on $S' = \{e'_0, e_1, \dots, e_n, e'_1, \dots, e'_t\}$; $e'_1, \dots, e'_t$ are elements parallel to e_0 and we suppose that S' contains no other parallel elements.

We eliminate all the elements parallel to e'_0 and define on $S = \{e_0 = e'_0, e_1, \dots, e_n\}$ a matroid M in the following way.

The circuits of M' which do not contain e'_0 become circuits in M . The dependent sets of M' which are not circuits in M' but contain e'_0 and are minimal in S , become circuits containing e_0 in M .

Let F be an e'_0 -flow in M' .

Corollary 3. *If F is extended on the set of the circuits of M which contain e_0 , so that condition (1) of [4] should be verified for $i \neq 0$, then F is an e_0 -flow in M .*

Proof. We define

$$l(e_0) = \sum_{i=1}^t l(e'_i) + l(e'_0),$$

$$L(e_0) = \sum_{i=1}^t L(e'_i) + L(e'_0)$$

and Theorem 2 and Corollary 2 assure us that F is an e_0 -flow in M .

REFERENCES

1. Ford L. R., Fulkerson D. R., *Flows in Networks*. Princeton Univ. Press, 1962.
2. Ghiuș V., *Circulație simplă într-un circuit de sculuri*. Lucrările Primului Simpozion Național de Inventică, Iași, 1984.
3. Ghiuș V., *Circulație simplă într-un circuit de circuite*. Lucrările Primului Simpozion Național de Inventică, Iași, 1984.
4. Lovinescu C., *Conditions for the Existence of Flow in a Matroid*. Bul. Inst. polit. Iași, Supliment „Cercetări matematice” (1985).
5. Welsh D. J. A., *Matroid Theory*. Academic Press, 1976.

FLUX ÎN MATROIZI CU ELEMENTE PARALELE

(Rezumat)

Pornind de la condiții de existență a fluxului într-un matroid simplu, stabilite în [4], se studiază existența acestuia în matroizi cu elemente paralele.

UNSTABLE MODES ON A BEAM-PLASMA SYSTEM

BY

CORNELIU D. CIUBOTARIU and VIOREL A. STANCU

1. Introduction. The interaction of an electron beam with a plasma is known to give rise to unstable plasma waves which can grow to relatively large amplitudes. It has been theoretically [1] and experimentally [2] [3] demonstrated that such plasmas are susceptible to parametric instability (for example, a four-mode interaction [4]) which may complete with trapping as a saturation mechanism.

In this paper we wish to investigate the latter mechanism on the basis of a multiple scale perturbation analysis of the Van der Pol equation.

2. Spatial Van der Pol Oscillator. When a low-density, cold electron beam is injected into a collisionless, Maxwellian plasma along a strong magnetic field, unstable waves in a narrow frequency band grow exponentially in space from the point where the beam is injected into the plasma. All these waves grow from the thermal noise level. The growth continues at the rate predicted by linear theory until the fastest-growing-wave's amplitude is sufficient to trap the beam electrons. At that point the wave's growth saturates (the wave amplitude stops growing), followed by a slow spatial amplitude oscillation about a mean value, as the trapped beam electrons slosh back and forth in the troughs of the wave. The oscillations result from an energy exchange between the wave and the beam electrons which are sloshing back and forth in the potential wells of the wave. Any wave within the unstable frequency band will exhibit this growth, saturation, and oscillation, provided that its initial amplitude is sufficiently large to allow that wave to trap the beam first. Thus, during its initial excitation, exponential growth and initial saturation stage, each unstable eigenmode may be regarded as a Van der Pol oscillator [5].

The spatially evolving dimensionless wave amplitude $\varphi_k(x)$ of each unstable mode with wave number k_n is modeled by Van der Pol equation [6]

$$(2.1) \quad \frac{d^2\varphi}{dx^2} - \varepsilon(1 - b\varphi^2) k_n \frac{d\varphi}{dx} + k_n^2 \varphi = 0,$$

where ε and b are dimensionless constants with

$$(2.2) \quad 0 < \varepsilon \ll 1, \quad \text{and} \quad b > 0.$$

For small-amplitude perturbations (before the nonlinear term $\varepsilon b \varphi^2 k_n \frac{d\varphi}{dx}$ becomes important), the beam-plasma system is linearly unstable with spatial linear growth rate

$$(2.3) \quad g = \frac{1}{2} \varepsilon k_n.$$

For example, in the case of the fastest-growing mode (maximum growth rate), the linear growth rate is [7]

$$(2.4) \quad g = \frac{4}{3} \frac{1}{2} \eta^{\frac{1}{3}} k_n \ll k_n, \quad \frac{g^{-1}}{k_n^{-1}} = \frac{2}{\varepsilon} \gg 1,$$

and the saturation amplitude $\varphi(x \rightarrow \infty, \text{ and } \varphi_s \sim \frac{m}{e} \left(u - \frac{\omega}{k_n} \right)^2$,

$$(2.5) \quad \varphi_s \equiv 2 b^{-\frac{1}{2}} \simeq \frac{m u^2}{e} (\eta)^{\frac{2}{3}},$$

where

$$\eta = \frac{1}{3} \frac{n_b}{n_0} \left(\frac{u}{v} \right)^2,$$

u is the beam velocity, $\bar{v}$ is the thermal velocity of the plasma electrons, $n_b/n_0 \ll 1$ is the ratio of the beam to plasma density and e/m is the ratio of electron charge to mass.

To take into account the discrepancy between the two space scales, oscillation space scale k_n^{-1} and growth space scale g^{-1} , we are led to use the well-known method of „multiple space scales“ of Krylov-Bogoliubov-Frieman-Sandri [8]...[15] for perturbation analysis of the Van der Pol equation. According to this method we extend the number of space variables from one variable x to many space variables $x_0, x_1, x_2, \dots$, where

$$(2.6) \quad \frac{dx_0}{dx} = 1, \quad \frac{dx_1}{dx} = \varepsilon, \quad \frac{dx_2}{dx} = \varepsilon^2, \dots,$$

so that we have for the derivatives d/dx and d^2/dx^2

$$(2.7) \quad \frac{d}{dx} = \frac{\partial}{\partial x_0} + \varepsilon \frac{\partial}{\partial x_1} + \varepsilon^2 \frac{\partial}{\partial x_2} + \dots,$$

$$(2.8) \quad \frac{d^2}{dx^2} = \frac{\partial^2}{\partial x_0^2} + \varepsilon \left(2 \frac{\partial^2}{\partial x_1 \partial x_0} \right) + \varepsilon^2 \left(\frac{\partial^2}{\partial x_1^2} + 2 \frac{\partial^2}{\partial x_2 \partial x_0} \right) + \dots$$

We expand $\varphi(x)$ in the small parameter ε according to

$$(2.9) \quad \varphi = \varphi^{(0)}(x_0, x_1, \dots) + \varepsilon \varphi^{(1)}(x_0, x_1, \dots) + \varepsilon^2 \varphi^{(2)}(x_0, x_1, \dots) + \dots$$

Substituting Eqs. (2.7)–(2.9) into Eq. (2.1), and equating to zero the coefficients of successive powers of ε , we calculate the successive approximations of the wave amplitude φ .

In order ϵ^0 , Eq. (2.1) becomes

$$(2.10) \quad \left(\frac{\partial^2}{\partial x_0^2} + k_n^2 \right) \varphi^{(0)}(x_0, x_1, \dots) = 0$$

and may be integrated on the fast spatial oscillation scale x_0 to give

$$(2.11) \quad \varphi^{(0)}(x_0, x_1, \dots) = A(x_1) \cos [k_n x_0 + \theta(x_1)],$$

where the amplitude $A(x_1)$, and phase $\theta(x_1)$ are allowed to vary on the slow (growth) space scale x_1 .

In order ϵ^1 , Eq. (2.1) becomes

$$(2.12) \quad \begin{aligned} \left(\frac{\partial^2}{\partial x_0^2} + k_n^2 \right) \varphi^{(1)}(x_0, x_1, \dots) = & -2 \frac{\partial^2}{\partial x_1 \partial x_0} \varphi^{(0)}(x_0, x_1, \dots) + \\ & + [1 - b \varphi^{(0)^2}(x_0, x_1, \dots)] k_n \frac{\partial}{\partial x_0} \varphi^{(0)}(x_0, x_1, \dots). \end{aligned}$$

Upon substitution of Eq. (2.11) into Eq. (2.12), we have

$$(2.13) \quad \begin{aligned} \left(\frac{\partial^2}{\partial x_0^2} + k_n^2 \right) \varphi^{(1)}(x_0, x_1, \dots) = & -[1 - b A^2 \cos^2(k_n x_0 + \theta)] k_n^2 A \sin(k_n x_0 + \theta) + \\ & + 2 k_n \frac{\partial A}{\partial x_1} \sin(k_n x_0 + \theta) + 2 k_n \frac{\partial \theta}{\partial x_1} A \cos(k_n x_0 + \theta), \end{aligned}$$

or, because $\cos^2 \alpha \sin \alpha = \frac{1}{4} (\sin \alpha + \sin 3\alpha)$,

$$(2.14) \quad \begin{aligned} \left(\frac{\partial^2}{\partial x_0^2} + k_n^2 \right) \varphi^{(1)}(x_0, x_1, \dots) = & 2 k_n \left[\frac{\partial A}{\partial x_1} - \frac{k_n}{2} A \left(1 - \frac{b A^2}{4} \right) \right] \sin(k_n x_0 + \theta) + \\ & + 2 k_n \frac{\partial \theta}{\partial x_1} A \cos(k_n x_0 + \theta) + \frac{1}{4} b A^3 k_n^2 \sin 3(k_n x_0 + \theta). \end{aligned}$$

The solution to this equation is

$$(2.15) \quad \varphi^{(1)}(x_0, x_1) = \varphi_{\text{secular}}^{(1)}(x_0, x_1) + \varphi_{\text{nonsec}}^{(1)}(x_0, x_1),$$

where

$$(2.16) \quad \begin{aligned} \varphi_{\text{secular}}^{(1)}(x_0, x_1) = & - \left[\frac{\partial A}{\partial x_1} - \frac{k_n}{2} A \left(1 - \frac{b A^2}{4} \right) \right] k_n \cos(k_n x_0 + \theta) + \\ & + \frac{\partial \theta}{\partial x_1} A \sin(k_n x_0 + \theta) \end{aligned}$$

and

$$(2.17) \quad \varphi_{\text{nonsec}}^{(1)}(x_0, x_1) = A_1 \cos(k_n x_0 + \theta) - \frac{b A^3}{32} \sin 3(k_n x_0 + \theta).$$

By cancelling the secularities on the fast spatial scale in the Eq. (2.16), we obtain

$$(2.18) \quad \frac{\partial A}{\partial x_1} = \frac{1}{2} k_n A (1 - \frac{1}{2} b A^2)$$

and

$$(2.19) \quad \frac{\partial \theta}{\partial x_1} = 0.$$

These differential equations determine the space development of $A(x_1)$ on a slower space scale x_1

$$(2.20) \quad A(x_1) = 2b^{-1/2} \left[1 + \left(\frac{4}{bA^2} - 1 \right) e^{-k_n x_1} \right]^{-1/2},$$

where A_0 is the amplitude in the point $x_1=0$ where the beam is injected into the plasma. Returning to the physical space variable x , the solution (2.11) takes the form.

$$(2.21) \quad \varphi(x) \simeq \varphi^{(0)}(x) = 2b^{-1/2} \left[1 + \left(\frac{4}{bA^2} - 1 \right) e^{-\epsilon k_n x} \right]^{-1/2} \cos(k_n x + \theta).$$

Thus, in the limit $\epsilon k_n x \ll 1$, Eq.(2.21) becomes

$$(2.22) \quad \varphi(x) \simeq A_0 \cos(k_n x + \theta).$$

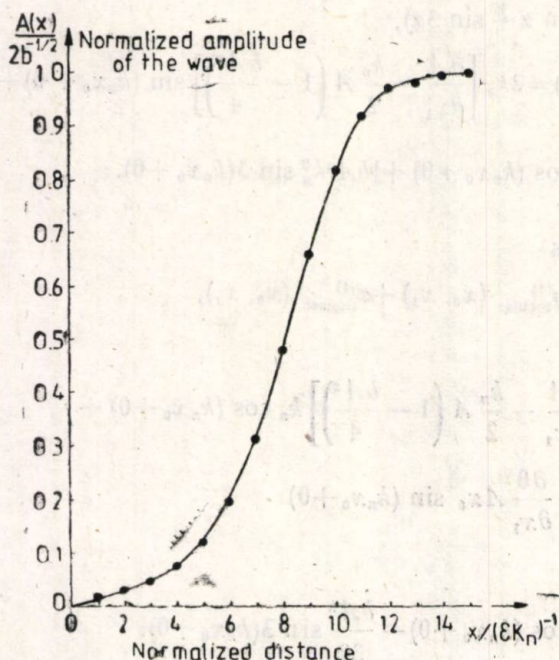


Fig. 1. — The case $A_0/(2b^{-1/2}) = 10^{-2} A(x)/(2b^{-1/2})$ increases to the saturation value.

From Eq. (2.18) we see that the amplitude of the oscillation grows when $A_0^2 < 4b^{-1}$, and saturates ($\partial A / \partial x_1 = 0$) when $A = 2b^{-1/2}$. If $\varepsilon k_n x \gg 1$, Eq. (2.21) becomes

$$(2.23) \quad \varphi(x) \simeq 2b^{-1/2} \cos(k_n x + \theta).$$

The variation of the normalized amplitude $A(x)/(2b^{-1/2})$ is illustrated in Fig. 1.

Consequently, the initial linear and nonlinear evolution of the unstable modes on a low-density, cold-electron beam-plasma system may be considered as an ensemble of Van der Pol oscillators. The nonlinear saturation amplitude $\varphi_s = 2b^{-1/2}$ depends analytically on all the basic parameters of the problem (i.e., plasma density, beam density, plasma thermal velocity, and beam drift velocity).

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REFERENCES

1. Nishikawa K., J. Phys. Soc. Jap., 24, 916 (1968).
2. Stenzel R., Wong A., Phys. Rev. Lett., 28, 274 (1972).
3. Porkolab M., Arunasalam V., Ellis R. Jr., Phys. Rev. Lett., 29, 1438 (1972).
4. Parker R. R., Throop A. L., Phys. Rev. Lett., 31, 1549 (1973).
5. DeNeef, Lashinsky H., Phys. Rev. Lett., 31, 1039 (1973).
6. Van der Pol B., Phil. Mag., 43, 700 (1922).
7. O'Neil T. M., Winfrey J. H., Malmberg J. H., Phys. Fluids, 15, 1514 (1972).
8. Krylov N., Bogoliubov N., *Introduction to Non-linear Mechanics*. Princeton Univ. Press, 1947.
9. Bogoliubov N., Mitropolskii I., *Les Méthodes Asymptotiques en Théorie des Oscillations Non-linéaires*. Gauthier-Villars, Paris, 1962.
10. Frieman A. E., J. Math. Phys., 4, 410 (1963).
11. Sandri G., Ann. Phys., 24, 332 (1963).
12. Sandri G., Nuovo Cimento, 36, 67 (1965).
13. Montgomery D., Tidman D. A., Phys. Fluids, 7, 242 (1964).
14. Cardinola P., de Barlieri O., Maroli C., Nuovo Cimento, 42B, 266 (1966).
15. Davidson R. C., *Methods in Non-linear Plasma Theory*. Academic Press, N Y, 1972.

MODURI INSTABILE ÎNTR-UN SISTEM FASCICUL-PLASMĂ

(Rezumat)

Este studiată interacțiunea fascicul-plasmă, în ipoteza că sistemul fascicul plasmă poate fi descris de o ecuație de tip Van der Pol.

from Eq. (2.18) we see that the amplitude of the oscillation grows when $\frac{1}{2} < \frac{1}{2} \frac{d}{dt} \ln \frac{1}{2} < 1$, and saturates when $\frac{1}{2} \frac{d}{dt} \ln \frac{1}{2} = 1$, Eq. (2.19) becomes

$$\frac{1}{2} \frac{d}{dt} \ln \frac{1}{2} = 1 \quad (2.20)$$

The variation of the normalised amplitude $\frac{1}{2} \frac{d}{dt} \ln \frac{1}{2}$ is illustrated in Fig. 1.

Consequently, the initial linear and nonlinear evolution of the system on a low-density, cold-electron beam-plasma system may be considered as an example of Van der Pol oscillators. The nonlinear saturation amplitude $\frac{1}{2} \frac{d}{dt} \ln \frac{1}{2} = 1$ depends qualitatively on all the parameters of the problem (i.e. plasma density, beam density, plasma thermal velocity, and beam drift velocity).

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REFERENCES

1. Krukar, R. J. *Phys. Rev. Lett.* **34**, 910 (1975).
2. Krukar, R. J. *Phys. Rev. Lett.* **35**, 974 (1975).
3. Krukar, R. J. *Phys. Rev. Lett.* **36**, 1000 (1976).
4. Krukar, R. J. *Phys. Rev. Lett.* **37**, 1000 (1976).
5. Krukar, R. J. *Phys. Rev. Lett.* **38**, 1000 (1977).
6. Krukar, R. J. *Phys. Rev. Lett.* **39**, 1000 (1977).
7. Krukar, R. J. *Phys. Rev. Lett.* **40**, 1000 (1978).
8. Krukar, R. J. *Phys. Rev. Lett.* **41**, 1000 (1978).
9. Krukar, R. J. *Phys. Rev. Lett.* **42**, 1000 (1979).
10. Krukar, R. J. *Phys. Rev. Lett.* **43**, 1000 (1979).
11. Krukar, R. J. *Phys. Rev. Lett.* **44**, 1000 (1980).
12. Krukar, R. J. *Phys. Rev. Lett.* **45**, 1000 (1980).
13. Krukar, R. J. *Phys. Rev. Lett.* **46**, 1000 (1981).
14. Krukar, R. J. *Phys. Rev. Lett.* **47**, 1000 (1981).
15. Krukar, R. J. *Phys. Rev. Lett.* **48**, 1000 (1982).

MODEL INSTABLE INTERACTING PARTICLE PLASMA

(Received)

The study of the interaction between a cold electron beam and a warm plasma is of interest in the context of the Van der Pol oscillator.

B.E.M. IN ELASTIC TORSION PROBLEMS USING GREEN'S GENERALIZED THEOREM AND R -FUNCTIONS

BY

VICTOR-FLORIN POTERAȘU and N. MIHALACHE

*Dedicated to Mr. Prof. Mangeron
for his jubilee year*

1. Introduction. The engineering problems (potential, elasto-static, elasto-dynamic, etc.) can be solved numerically by means of Green's generalized theorem. For use the Boundary Element Methods it is necessary to determine the fundamental solutions and for to describe the complex boundaries in the real domain the better is the introduction of the algebraic logical functions called R -functions. These functions lead to an analytical-numerical method that has been successfully applied to various boundary value problems of mathematical physics for differential partial equations. The R -functions allows an increasing of accuracy in BEM for the complex problems and it is not necessary a transform mapping with the singularity in the imaginary plane.

In the paper we present a method for to construct a quasi-function Green based on the Green's generalized theorem and R -function for the Poisson equation with Dirichlet, Neumann boundary conditions. The solution is given for two and three dimensional cases. Finally, we present a practical application for the elastic torsion plane problems.

2. B.E.M. in Elastic Problems Using Green's Generalized Theorem. Green's generalized theorem lies certainly at the heart of the BEM; its formulation for the laplacean operator is the standpoint for the BEM in potential theory, while from its generalized form BEM can be derived in elastic problems. As it is well known, Green's theorem in all its formulations is an application of Gauss or divergence theorem, which for any vector u states that (Brebba, Futagami, Tanaka, 1983)

$$(1) \quad \int_{\Omega} \operatorname{div} \tilde{u} \, d\Omega = \int_{\Gamma} \tilde{u} \tilde{n} \, d\Gamma,$$

where n is the outward normal on the boundary Γ of the domain Ω . In order to apply Gauss' theorem to a general differential operator L acting on a vector u , the differential operator has to be transformed into a di-

vergence expression. The key to perform this step is the adjoint differential operator L^* of L which is defined as follows

$$(2) \quad \tilde{v} \tilde{L} \tilde{u} - \tilde{u} \tilde{L}^* \tilde{v} = \operatorname{div} \tilde{p},$$

where $\tilde{v}$ is a vector on which $\tilde{L}$ operates and $\tilde{p}$ is a suitable vector. By integrating Eq. (2) over the domain Ω and by making use of Gauss' theorem (1), the generalized form of Green's theorem is obtained (Kobayashi, Nishimura, 1980)

$$(3) \quad \int_{\Omega} (\tilde{v} \tilde{L} \tilde{u} - \tilde{u} \tilde{L}^* \tilde{v}) d\Omega = \int_{\Gamma} \tilde{p} n d\Gamma.$$

A further extension of Green's theorem can be obtained if we consider three different vectors $\tilde{v}$, i.e. the dyadic $\tilde{v}' = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$

$$(4) \quad \int_{\Omega} (\tilde{v}' \tilde{L} \tilde{n} - \tilde{n} \tilde{L}^* \tilde{v}') d\Omega = \int_{\Gamma} \tilde{p}' n d\Gamma,$$

where $\tilde{p}' = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$.

We will now apply Green's generalized theorem in elastostatics under the assumptions of both geometrical and material linearity. The governing equations can be expressed as follows:

$$(5) \quad \begin{aligned} (a) \quad & \tau_{ij,j} + b_i = 0, \\ (b) \quad & \varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \\ (c) \quad & \tau_{ij} = C_{ijkl} \varepsilon_{kl}, \end{aligned}$$

where C_{ijkl} is the elasticity tensor and (Poterășu, Mihalache, 1985)

$$(6) \quad C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}.$$

By substituting Eq. (5b) in (5c) and then (5c) in (5a) we obtain

$$(7) \quad C_{ijkl} u_{k,jl} + b_i = 0,$$

where Eq. (7) is Navier-Cauchy's generalized equations for the hyperelastic medium. In elastostatic problems we use the boundary conditions of Dirichlet type $u_i = \bar{u}_i$ and Neumann type (Fig. 1)

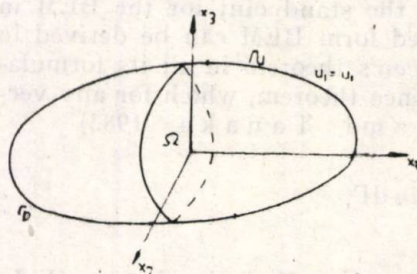


Fig. 1. — Dirichlet and Neumann boundary conditions.

$$(8) \quad C_{ijkl} u_{k,l} n_j = \bar{p}_i.$$

For to use the Green's generalized theorem we determine the Navier-Cauchy selfadjoint operator

$$(9) \quad \Delta = C_{ijkl} \frac{\partial^2}{\partial x_j \partial x_l}$$

and integrating two times by parts it results

$$(10) \quad C_{ijkl}(v_i u_{k,jl} - u_i v_{k,jl}) = C_{ijkl} \frac{\partial}{\partial x_j} (v_i u_{k,l} - u_i v_{k,l}).$$

In that case Eq. (3) becomes

$$(11) \quad \int_{\Omega} C_{ijkl}(v_i u_{k,jl} - u_i v_{k,jl}) d\Omega = \int_{\Gamma} C_{ijkl}(v_i u_{k,l} - u_i v_{k,l}) n_j d\Gamma,$$

which can be written in a more familiar form by making use of the relations of the same kind of (8) between tractions $\tilde{p}(\tilde{u})$, $\tilde{p}(\tilde{v})$ on the boundary and the displacements $\tilde{u}$, $\tilde{v}$

$$(12) \quad \int_{\Omega} C_{ijkl}(v_i u_{k,jl} - u_i v_{k,jl}) d\Omega = \int_{\Gamma} [v_i p_j(u) - u_i p_j(\tilde{v})] d\Gamma.$$

The next step is to transform Green's identity (11) into a boundary integral representation of the displacement $\tilde{u}$. If is considered the displacement $\tilde{v}(\tilde{x}, \tilde{\xi})$ of a generical point $\tilde{x}$ due to a unit concentrated force acting within an infinite body at a given point $\tilde{\xi}$ in a given direction, as a Green's vector function is associated with each unit force a Green's dyadic function $v_{im}(x, \xi)$ is thus obtained, where $m=1, 2, 3$ shows the direction of the force. Eq. (11) has to be put also in vector form as follows

$$(13) \quad \int_{\Omega} C_{ijkl}(v_{im} u_{k,jl} - u_i v_{k,m,jl}) d\Omega = \int_{\Gamma} C_{ijkl}(v_{im} u_{k,l} - u_i v_{k,m,l}) n_j d\Gamma.$$

Eq. (13) could have been derived directly from Green's generalized theorem in its vector form (14). Green's dyadic function $v_{km}(x, \xi)$ can be regarded as the solution of the three adjoint problems

$$(14) \quad C_{ijkl} v_{km,jl} + \delta_{im}(x, \xi) = 0,$$

where $\delta_{im}(x, \xi)$ is a dyadic Dirac delta function.

The BEM is founded on an integral representation of the unknown function, i.e. the Green's formula or, more specifically for the elastostatic problem, the Somigliana identity. This is quite evident for the derivation from Betti's theorem; the reciprocity theorem can be obtained immediately from Eq. (12) [1]

$$(15) \quad \int_{\Omega} (u_i b_i - v_i b_i) d\Omega = \int_{\Gamma} [v_i p_i(u) - u_i p_i(v)] d\Gamma.$$

For the structural elements torsioned if we consider 2 points, M and N , on the meridian curve we obtain by successive integrations of Eq. (15)

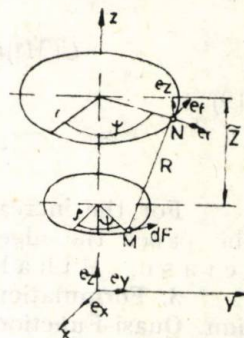


Fig. 2 — Axisymmetrical torsion problem.

$$(16) \quad C(N)u(M) + 2\pi \int_{\Gamma} P(M, N)u(N)r(N) d\Gamma = 2\pi \int_{\Gamma} Q(M, N)p(N)r(N) d\Gamma.$$

For the axisymetrical torsion problem the fundamental solutions $P(M, N)$ and $Q(M, N)$ are (Fig. 2)

$$(17) \quad P(M, N) = \frac{1}{4\pi^2 \rho^2 e^2} \left[\frac{3}{2} \sqrt{(e+\rho)^2 + z^2} n_r + \frac{e^2 + \rho^2 + \bar{z}^2}{((e-\rho)^2 + z^2) \sqrt{(e+\rho)^2 + \bar{z}^2}} \left((\rho^2 - e^2 + \bar{z}^2) \frac{n_r}{2} - e \bar{z} n_z \right) \right] E\left(\frac{\pi}{2}, \kappa\right) + \frac{e \bar{z} n_z - (e^2 + 2\rho^2 + 2\bar{z}^2) n_r}{V(e+\rho)^2 + \bar{z}^2} k\left(\frac{\pi}{2}, \kappa\right),$$

$$Q(M, N) = \frac{1}{4\pi^2 \rho^2 e G} \left\{ \frac{e^2 + \rho^2 + \bar{z}^2}{V(e+\rho)^2 + \bar{z}^2} k\left(\frac{\pi}{2}, \kappa\right) - V(e+\rho)^2 + \bar{z}^2 E\left(\frac{\pi}{2}, \kappa\right) \right\},$$

where

$$G = \sqrt{\frac{4e\rho}{(e+\rho)^2 + \bar{z}^2}}, \quad k\left(\frac{\pi}{2}, \kappa\right) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \kappa^2 \sin^2 \psi}},$$

$$E\left(\frac{\pi}{2}, \kappa\right) = \int_0^{\pi/2} \sqrt{1 - \kappa^2 \sin^2 \psi} d\psi.$$

The accuracy of the solution is determined by the interpolation functions. Brebbia proposed the quadratic functions

$$(18) \quad \psi_1(\xi) = \frac{1}{2}\xi(\xi-1), \quad \psi_2(\xi) = 1 - \xi^2, \quad \psi_3(\xi) = \xi(\xi+1).$$

We can obtain the discretized version of the integral equation as follows

$$(19) \quad C(M^q)u(M^q) + \sum_{l=1}^n \sum_{m=1}^q u_{ml} \int P(M^q, \xi) \psi_m(\xi) J_e(\xi) 2\pi r(\xi) d\xi = \sum_{l=1}^n \sum_{m=1}^q P_{ml} \int u(M^q, \xi) \psi_m(\xi) J_e(\xi) 2\pi r(\xi) d\xi.$$

For the increasing of accuracy in the boundary solutions we use in the paper the algebraic-logical R -functions (R v a c h e v, 1982; P o t e r a ș u, M i h a l a c h e, 1985).

3. Formulation of the B.E.M. by R -Functions for the Poisson Equation. Quasi-Function Green. The integrals from Eq. (19) depend on the interpolation function (18). Such solution may be obtained by many approximate methods like BEM and also, as we shall show, by the R -functions method. For a domain let us consider a boundary value problem given in the local formulation (O r k i s z, R a c h o w i c z, W n u k, 1984)

$$(20) \quad \begin{aligned} Au &= f \quad \text{in } \Omega \subset \mathbb{R}^n \quad u \in x(\Omega), \\ l_i u_i &= \Phi_i \quad \text{on } \Gamma_i \subset \Gamma, \quad i=1, 2, \dots, m, \quad \bigcup_{i=1}^m \Gamma_i = \Gamma, \end{aligned}$$

or in the equivalent global representation

$$(21) \quad \int_{\Omega} k(u) \, d\Omega + \int_{\Gamma} G(u) \, d\Gamma = \min_{\text{u}}$$

Here A, K, l, G are operators defined over a space of functions defined in the domain Ω and on Γ respectively. The general structure of the solution of the problem is a mapping $B: m \rightarrow x(\Omega)$ that satisfies the following conditions

$$(22) \quad l_i B(\Phi) = \Phi_i \quad \text{on } \Gamma_i, \quad i=1, 2, \dots, m \quad \forall \Phi \in m,$$

in which m is a function space defined over Ω . Taking this into account we may present the general structure of solution in a more specific form $B = B(\Phi, w, \Phi)$ where $w: \mathbb{R} \rightarrow \mathbb{R}$ is a mapping that defines the domain

$$(23) \quad \Omega = \{x \in \mathbb{R}^n: w(x) > 0\}, \quad \Gamma = \{x \in \mathbb{R}^n: w(x) = 0\}$$

and $\Phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is a mapping defined over the domain Ω , that strictly satisfies all prescribed boundary conditions. If we consider the Poisson's equation with the Dirichlet boundary conditions, we can write

$$(24) \quad \Delta v \equiv \sum_{j=1}^n \frac{\partial^2 v}{\partial x_j^2} = f_0 \quad x \in \Omega \quad v|_{\Gamma} = \varphi_0,$$

where $\varphi_0: \Sigma_0 \rightarrow \mathbb{R}^m$, $L \subset \Sigma_0$ is an elementary function of a set functions φ on the domain Ω and $\varphi_{\Gamma} = \varphi_0$. If we denote $u = v - \varphi$ we get

$$(25) \quad \Delta u = f, \quad x \in \Omega, \quad u|_{\Gamma} = 0,$$

where $f = f_0 - \Delta \varphi$.

Applying the Green's formula for all the functions $u, q \in C^{\Omega}(\Omega \cap \Gamma)$ it results

$$(26) \quad u(x) = \frac{1}{2\pi} \int_{\Gamma} \left[\frac{\partial u(\xi)}{\partial n} \ln \frac{1}{r} - u(\xi) \frac{\partial}{\partial n} \ln \frac{1}{r} \right] d_{\xi} \Gamma - \frac{1}{2\pi} \int_{\Omega} \ln \frac{1}{r} \Delta \varphi d_{\xi} \Omega.$$

Integrating (26) we can write

$$(27) \quad \int_{\Omega} [u(\xi) \Delta_{\xi} q(\xi) - q(\xi) \Delta_{\xi} u(\xi)] d\xi = \int_{\Gamma} \left[u(\xi) \frac{\partial q(\xi)}{\partial n} - q(\xi) \frac{\partial u(\xi)}{\partial n} \right] d_{\xi} \Gamma,$$

where

$$r = \|x - \xi\| = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}, \quad \Delta \xi = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}.$$

In the relation (27) q has the form

$$(28) \quad q = q(x, \xi) = -\frac{1}{2} \ln [r^2 + 4w(x)w(\xi)],$$

defined $q(x, \xi) \in C^2(\Omega \times \Omega)$ and $w=0$ for $x \in \Gamma$.

If $G_2(x, \xi) = \ln(1/r) - q(x, \xi)$ and $G_2(x, \xi)|_{w(x)=0} = 0$ it results

$$(29) \quad u(x) = u_0(x) + \int_0^x u(\xi) K'(x, \xi) d\xi,$$

where

$$(30) \quad u_0(x) = -\frac{1}{2\pi} \int_0 G_2(x, \xi) f(\xi) d\xi, \quad K'(x, \xi) = -\frac{1}{2\pi} \Delta_{\bar{z}} q(x, \xi),$$

If $x=\xi$ the expression $K'(x, \xi)$ becomes

$$(31) \quad K'(x, \xi)|_{x=\xi} = \frac{1}{2\pi} \frac{1 + w\Delta w - (\nabla w)^2}{w^2}.$$

But, for $w=0$ we can write $(\Delta w)^2=1+w\Phi$ with $\Phi \in C(\Omega \cup \Gamma)$ and

$$(32) \quad K'(x, \xi) \big|_{x=\xi} = \frac{1}{2\pi} \frac{\Delta w - \Phi}{w}.$$

When in relation (32) $w \rightarrow 0$

$$(33) \quad \Delta w|_{w=0} = \Phi|_{w=0} = \frac{(\Delta w)^2 - 1}{w} \Big|_{w=0}.$$

For a function $w_0=0$ on Γ it results $\Phi_0=[(\Delta w_0)^2-1]w_0^{-1}$, thus the function w becomes

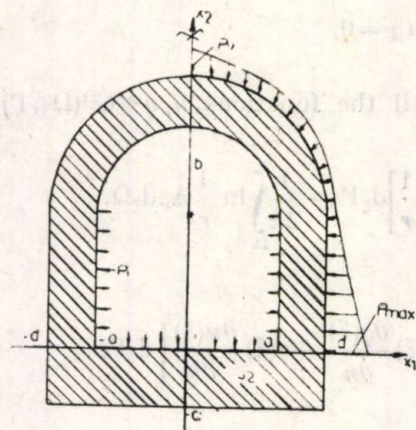


Fig. 3 = Water channel section.

$$(34) \quad w = w_0 + w_0^2 \psi,$$

and $\psi = \frac{1}{2}(\Delta w_0 - \Phi_0)$.

In an extensive form

$$(35) \quad w = w_0 + \frac{1}{2} w_0^2 (\Delta w_0 - \Phi_0) = w_0 + \frac{1}{2} w_0^2 \left[\Delta w_0 - \frac{(\nabla w_0)^2 - 1}{w_0} \right].$$

The algebraic-logical operator w can be written for the usual domains and if introduced in the relation (19) it leads to the increasing of the solution accuracy. For the water channel with the form shown in Fig. 3 the function w is represented by

$$w_1 \equiv \frac{1}{2a} (a^2 - x_1^2) \geq 0, \quad x_1 = \pm a, \quad w_2 \equiv (x_2 - b), \quad x_2 = b;$$

$$\begin{aligned}
 w_3 &\equiv (x_2 + c) \geq 0, \quad x_2 = -c; & w_4 &\equiv \frac{1}{2d} (d^2 - x_1^2), \quad x_1 = \pm d; \\
 (36) \quad w_5 &\equiv \left\{ \frac{1}{2a} [a^2 - x_1^2 - (x_2 - b)^2] \geq 0 \right\} & & \text{(circle } r=a; O_1(0, b)); \\
 w_6 &\equiv \left\{ \frac{1}{2d} [d^2 - x_1^2 - (x_2 - b)^2] \geq 0 \right\} & & \text{(circle } r=d; O(0, b)); \\
 w_7 &\equiv x_2 \geq 0.
 \end{aligned}$$

The domain Ω has the form

$$(37) \quad \Omega = \{ [w_3 \wedge \bar{w}_2] \wedge w_4 \} \vee \{ w_6 \} \wedge \{ [w_7 \wedge \bar{w}_2] \wedge w_1 \} \vee w_5.$$

In the relation (19) for the boundary discretized we can introduce the functions w_i ($i=1, \dots, 7$) in order to describe the boundary with accuracy in the real domain.

Now, we will write the R -functions for the structural element torsioned in Fig. 4:

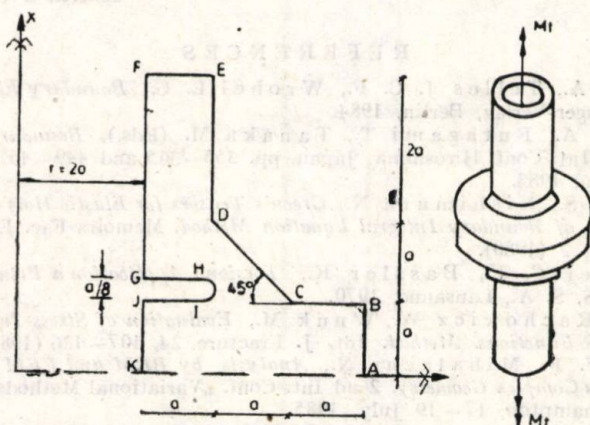


Fig. 4 Structural element torsioned.

$$\begin{aligned}
 w_1 &\equiv (5a - x_1) \geq 0, \quad \overline{AB}; & w_2 &\equiv (x_2 - a) \geq 0, \quad \overline{BC}; \\
 (38) \quad w_3 &\equiv \left(-\frac{\sqrt{2}}{2} x_1 - \frac{\sqrt{2}}{2} x_2 + \sqrt{2} \right) \geq 0, \quad \overline{DC}; & w_4 &\equiv (3a - x_1) \geq 0, \quad \overline{DE}; \\
 w_5 &\equiv (4a - x_2) \geq 0, \quad \overline{EF}; & w_6 &\equiv (2a - x_1) \geq 0, \quad \overline{FK}; \\
 w_7 &\equiv \left\{ \left(\frac{9a}{8} - x_2 \right) \vee \frac{1}{6a} [4a^2 - x_2^2 - (x_1 - 2a)^2] \vee (a - x_2) \right\} \geq 0, \quad \overline{GHIJ}; \\
 \text{and} & & w &\equiv \{ [w_1 \wedge w_2 \wedge w_3 \wedge w_4 \wedge w_5 \wedge w_6] \wedge w_7 \}.
 \end{aligned}$$

Interpolation functions ψ_m from (19) can be replaced with w_i from (38).

The discretized boundary from Fig. 4 can be introduced in a computer program in order to obtain a better accuracy of the stress state in a structural torsioned element.

4. Conclusions. 1. The R -functions method is more advantageous than another method for studying the stress, strain state in a body with complex form. This method avoids the mapping in a complex domain, conversion in a real domain.

2. The method was proved to be effective in elasticity problems, especially when fully computerized through the use of symbolic programming. Also, the method should give good results with few degrees of freedom.

3. The present paper may be considered as an initial step in the development of this novel analytical/numerical technique for the torsion problems and boundary element methods. The quasi-functions Green, the Green's generalized theorem for the Poisson's equation allow the torsional study by BEM.

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REFERENCES

1. Brebbia C. A., Telles J. C. F., Wrobel L. C., *Boundary Element Techniques*. Springer-Verlag, Berlin, 1984.
2. Brebbia C. A., Futagami T., Tanaka M. (Eds.), *Boundary Elements*. Proc. 5-th Int. Conf. Hiroshima, Japan, pp. 355–365 and 449–457, Springer-Verlag, Berlin, 1983.
3. Kobayashi S., Nishimura N., *Green's Tensors for Elastic Half-spaces. An Application of Boundary Integral Equation Method*. Memoirs Fac. Eng. Kyoto Univ., XLII, 2 (1980).
4. Kollbrunner C. F., Bassler K., *Torsion: Application a l'etude des structures*. SPES, S. A., Lausanne, 1970.
5. Orkisz J., Rachowicz W., Wnuk M., *Evaluation of Stress Intensity Factors by the R-Functions Method*. Int. J. Fracture, 24, 107–126 (1984).
6. Poterașu V. F., Mihalache N., *Analysis by BEM and FEM of a Plane Model with a Complex Geometry*. 2 nd Int. Conf. „Variational Methods in Engineering“, Southampton, 17–19 July, 1985.
7. Poterașu V. F., Mihalache N., *BEM and N-Functions Method for Design of Cross Pipelines with Complex Form*. 5-th Int. Symp. Freight Pipelines, Philadelphia, Pennsylvania, 14–16 Oct. 1985 (in print J. of Pipelines).
8. Rvachev V. A., *Theory of R-Functions and Some of its Applications* (in Russian). Kiev, Naukova Dumka, 1982.

METODA ELEMENTELOR DE FRONTIERĂ ÎN PROBLEMELE DE TORSIUNE ELASTICĂ FOLOSIND TEOREMA GENERALIZATĂ A LUI GREEN ȘI FUNCȚIILE R

(Rezumat)

Se prezintă o metodă pentru construcția cvasi-funcției Green bazată pe teorema generalizată a lui Green și funcția R pentru ecuația lui Poisson cu condiții pe contur Dirichlet și Neumann. Soluția este prezentată pentru cazurile bi și tridimensional. În final se prezintă aplicații practice pentru problemele de torsiune plane. Funcțiile algebrico-logice R pot îmbunătăți precizia soluției în metoda elementelor de frontieră aplicată corpurilor cu geometrie complexă.

SUR L'INVARIABILITÉ DE L'AXE INSTANTANÉ DE ROTATION

III. LE CAS DES DEUX REPERES TRIORTHOGONAUX,
L'UN DE CES DEUX REPERES SE MOUVANT EXCLUSIVEMENT PAR
UNE ROTATION PAR RAPPORT À L'AUTRE

UNE CONDITION NÉCESSAIRE POUR L'INVARIABILITÉ
DE L'AXE INSTANTANÉ DE ROTATION

PAR

C. ACKER

On a souligné dans [6] que lorsque le repère cartésien triorthogonal $X_1O_1Y_1Z_1$ est solidaire d'un corps rigide et le repère cartésien triorthogonal $X_2O_2Y_2Z_2$ a un mouvement quelconque par rapport au repère $X_1O_1Y_1Z_1$ (donc les deux repères mobiles n'étant pas solidaires l'un de l'autre et le mouvement de $X_2O_2Y_2Z_2$ par rapport au repère $X_1O_1Y_1Z_1$ n'étant pas réduit à une translation), alors les coefficients des vecteurs unités i_1, j_1, k_1 , dans les expressions analytiques des vecteurs unités i_2, j_2, k_2 (des axes du repère $X_2O_2Y_2Z_2$), par rapport au repère $X_1O_1Y_1Z_1$, sont des fonctions qui dépendent effectivement du temps t . Cela a impliqué (en considérant les mouvements des deux repères mobiles par rapport au repère fixe, aussi que le mouvement de l'un de ces deux repères mobiles par rapport à l'autre) les relations

$$(1) \quad \mathbf{w}_1 = \mathbf{w}_{12} + \mathbf{w}_2$$

et

$$(2) \quad \mathbf{w}_2 = \mathbf{w}_{21} + \mathbf{w}_1;$$

(1) et (2) impliquant la relation

$$(3) \quad \mathbf{w}_{12} = -\mathbf{w}_{21}$$

($\mathbf{w}_{12}$ représentant le vecteur vitesse angulaire du repère $X_1O_1Y_1Z_1$, lorsque le mouvement de ce repère est considéré par rapport au repère $X_2O_2Y_2Z_2$ — donc pour l'observateur solidaire du repère $X_2O_2Y_2Z_2$ —, tandis que $\mathbf{w}_{21}$ est le vecteur vitesse angulaire du repère $X_2O_2Y_2Z_2$ lorsqu'on considère le mouvement de ce repère par rapport au repère $X_1O_1Y_1Z_1$ — donc pour l'observateur solidaire du repère $X_1O_1Y_1Z_1$ — ; rappelons nous aussi que $\mathbf{w}_1$ est le vecteur vitesse angulaire pour le mouvement du repère $X_1O_1Y_1Z_1$,

par rapport au repère fixe, tandis que w_2 est le vecteur vitesse angulaire pour le mouvement du repère $X_2O_2Y_2Z_2$, par rapport au repère fixe).

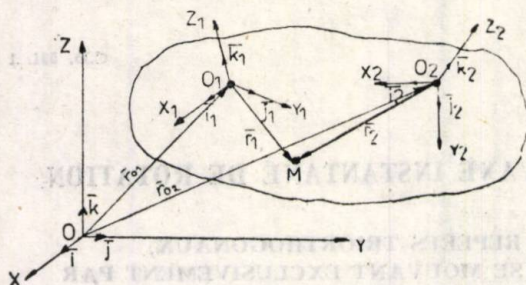


Fig. 1

Donc si le mouvement du repère cartésien triorthogonal $X_2O_2Y_2Z_2$, par rapport au repère cartésien triorthogonal $X_1O_1Y_1Z_1$ (qui est solidaire du corps rigide) a pour mouvement composant un mouvement de rotation, alors les dérivées des vecteurs unités i_2, j_2, k_2 par rapport au temps t , relatives au repère $X_1O_1Y_1Z_1$, ne sont pas nulles, ce qui implique

$$(4) \quad w_{12} = -w_{21} \neq 0$$

et alors les relations (1), (2) peuvent être considérées ayant leurs formes complètes (rappelons que si $X_2O_2Y_2Z_2$ se meut par rapport à $X_1O_1Y_1Z_1$ par une translation ([6], (11)), alors les relations (1) et (2) se réduisent à $w_1 = w_2$), donc

$$(5) \quad w_1 \neq w_2.$$

Alors les directions des axes instantanés de rotation des deux repères $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$, pendant les mouvements de ces repères par rapport au repère fixe, sont distinctes, ce qui implique — évidemment — que même les axes instantanés de rotation de ces deux repères mobiles, correspondants aux mouvements de ces repères par rapport au repère fixe, sont distincts — toujours lorsque le mouvement du repère $X_2O_2Y_2Z_2$ par rapport au repère $X_1O_1Y_1Z_1$ a pour mouvement composant une rotation. Par exemple, lorsque le repère $X_2O_2Y_2Z_2$ se meut, par rapport au repère $X_1O_1Y_1Z_1$, exclusivement par une rotation autour de l'origine O_2 — qui a une position fixe par rapport au repère $X_1O_1Y_1Z_1$ — alors, en général, on a $w_1 \neq w_2$.

En poursuivant donc le problème de l'invariabilité de l'axe instantané de rotation pendant la substitution du repère $X_1O_1Y_1Z_1$ par $X_2O_2Y_2Z_2$ (ou vice versa), lorsque O_2 a une position fixe par rapport au repère $X_1O_1Y_1Z_1$, observons que la coïncidence des deux axes instantanés de rotation des deux repères mobiles implique, dès le début, la coïncidence des directions de ces axes, c'est à dire implique

$$(6) \quad w_2 = \mu w_1,$$

en devant compléter, après, cette condition, tel que même les axes instantanés de rotation soient coïncidents.

Pour cela, considérons la vitesse par rapport au repère fixe, de l'origine O_2 du repère $X_2O_2Y_2Z_2$, exprimée par l'utilisation du repère $X_1O_1Y_1Z_1$ (qui est solidaire du corps rigide), (v. fig. 1)

$$(7) \quad v_{O_2} = v_{O_1} + w_1 \times \overrightarrow{O_1O_2}.$$

Alors, le vecteur différence

$$(8) \quad \mathbf{D} = \left(\mathbf{r}_{0_1} + \frac{\mathbf{w}_2 \times \mathbf{v}_{0_1}}{w_2^2} \right) - \left(\mathbf{r}_{0_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{0_1}}{w_1^2} \right)$$

devient (v. (6))

$$(9) \quad \mathbf{D} = \mathbf{r}_{0_1} - \mathbf{r}_{0_1} + \frac{\mathbf{w}_1}{w_1^2} \times \left(\frac{1}{\mu} \mathbf{v}_{0_1} - \mathbf{v}_{0_1} \right)$$

et alors (7) implique (v. aussi la fig. 1)

$$(10) \quad \mathbf{D} = \vec{O_1 O_2} + \frac{\mathbf{w}_1}{w_1^2} \times \left(\frac{1}{\mu} \mathbf{v}_{0_1} + \frac{1}{\mu} \mathbf{w}_1 \times \vec{O_1 O_2} - \mathbf{v}_{0_1} \right),$$

donc

$$(11) \quad \mathbf{D} = \vec{O_1 O_2} + \left(\frac{\mathbf{w}_1}{w_1^2} \times \mathbf{v}_{0_1} \right) \left(\frac{1}{\mu} - 1 \right) + \frac{\mathbf{w}_1}{\mu} \left(\frac{\mathbf{w}_1}{w_1^2} \cdot \vec{O_1 O_2} \right) - \frac{1}{\mu} \vec{O_1 O_2},$$

ou

$$(12) \quad \mathbf{D} = \left(1 - \frac{1}{\mu} \right) \left(\vec{O_1 O_2} - \frac{\mathbf{w}_1}{w_1^2} \times \mathbf{v}_{0_1} \right) + \frac{1}{\mu} \mathbf{w}_1 \left(\frac{\mathbf{w}_1}{w_1^2} \cdot \vec{O_1 O_2} \right).$$

L'équation vectorielle

$$(13) \quad \mathbf{r} = \mathbf{r}_{0_1} + \frac{\mathbf{w}_2 \times \mathbf{v}_{0_1}}{w_2^2} + \lambda \mathbf{w}_2$$

de l'axe instantané de rotation du repère $X_2 O_2 Y_2 Z_2$, correspondant au mouvement de ce repère par rapport au repère fixe — cette équation elle-même étant rapportée au repère fixe — devient (v. (6), (8), (12))

$$(14) \quad \mathbf{r} = \mathbf{r}_{0_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{0_1}}{w_1^2} + \left(1 - \frac{1}{\mu} \right) \left(\vec{O_1 O_2} - \frac{\mathbf{w}_1 \times \mathbf{v}_{0_1}}{w_1^2} \right) + \left[\frac{1}{\mu} \left(\frac{\mathbf{w}_1}{w_1^2} \cdot \vec{O_1 O_2} \right) + \lambda \mu \right] \mathbf{w}_1.$$

Donc si $w_1 \neq 0$ (nous excluons $w_1 = 0$, car si cette condition serait satisfaite, alors nous n'aurions pas de quoi chercher l'invariabilité, la direction de l'axe instantané de rotation du repère $X_1 O_1 Y_1 Z_1$ étant celle de $\mathbf{w}_1$, lorsqu'on considère le mouvement de ce repère par rapport au repère fixe) et $\mu \neq 1$, alors la coïncidence des deux axes instantanés de rotation, que nous poursuivons, implique aussi que le vecteur

$$(15) \quad \left(1 - \frac{1}{\mu} \right) \left(\vec{O_1 O_2} - \frac{\mathbf{w}_1 \times \mathbf{v}_{0_1}}{w_1^2} \right)$$

soit parallèle à $\mathbf{w}_1$, ou qu'il soit nul, c'est à dire

$$(16) \quad \left(1 - \frac{1}{\mu} \right) \left(\vec{O_1 O_2} - \frac{\mathbf{w}_1 \times \mathbf{v}_{0_1}}{w_1^2} \right) \times \mathbf{w}_1 = 0,$$

ou

$$(17) \quad \left(1 - \frac{1}{\mu} \right) \left(\vec{O_1 O_2} - \frac{\mathbf{w}_1 \times \mathbf{v}_{0_1}}{w_1^2} \right) = \nu \mathbf{w}_1, \quad (\nu \in \mathbb{R});$$

donc si le vecteur

$$(18) \quad \mathbf{W} = \frac{\mathbf{w}_1 \times \mathbf{v}_{0_1}}{w_1^2}$$

a son point d'application dans l'origine O_1 et par sa pointe passe la droite (Δ) , parallèle à w_1 (qui est perpendiculaire sur W), alors la position de l'origine O_2 (qui est fixe par rapport au repère $X_1O_1Y_1Z_1$) doit appartenir à (Δ) , lorsque les axes instantanés de rotation, correspondants aux mouvements des repères $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$ par rapport au repère fixe, coïncident (v. la fig. 2).

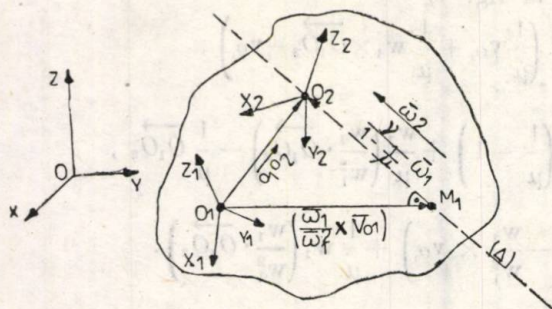


Fig. 2

Pour conclure, lorsque O_2 a une position fixe par rapport au repère $X_1O_1Y_1Z_1$, alors pour $\mu \neq 1$, il est nécessaire que $w_2 = \mu w_1$ (v. (6)) et que la droite (Δ) , parallèle à w_1 , qui passe par la pointe M_1 du vecteur W dont le point d'application est O_1 (c'est à dire l'axe instantané de rotation du repère $X_1O_1Y_1Z_1$ correspondant au mouvement de ce repère par rapport au repère fixe), passe

aussi toujours par O_2 (ce qui relève aussi que la position de M_1 — la pointe du vecteur W dont le point d'application est O_1 — doit appartenir toujours à une sphère dont un de ses diamètres est O_1O_2 (v. la fig. 2), cette sphère ayant, évidemment, sa position fixe par rapport au repère $X_1O_1Y_1Z_1$).

Dans ces conditions, les axes instantanés de rotation correspondants aux mouvements des repères $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$, par rapport au repère fixe sont coïncidents.

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BIBLIOGRAPHIE

1. Vîlcovici V., Bălan Șt., Voinea R., *Mecanică teoretică*. Ed. tehn., Buc., 1968.
2. Mangeron D., Irimescu N., *Mecanica rigidelor cu aplicații în inginerie*. Vol. 1, *Mecanica rigidului*. Ed. tehn., Buc., 1978.
3. Mangeron D., Irimescu N., *Mecanica rigidelor cu aplicații în inginerie*. Vol. 2, *Mecanica sistemelor de rigide*. Ed. tehn., Buc., 1980.
4. Acker C., *Elemente de mecanică tehnică. Introducere în mecanica rigidului*. Inst. polit. Iași, 1982.
5. Acker C., *Sur l'invariabilité de l'axe instantané de rotation (I)*. Bul. Inst. polit. Iași, XXX (XXXIV), 1-4, s. I, 19-24 (1984).
6. Acker C., Acker E.I., *Sur l'invariabilité de l'axe instantané de rotation (II)*. Bul. Inst. polit. Iași, XXXI (XXXV), 1-4, s. I, 7-12 (1985).

ASUPRA INVARIANȚEI AXEI INSTANTANEE DE rotație A RIGIDULUI (III)

(Rezumat)

Se stabilește o condiție necesară, cu caracter general, pentru coincidența axelor instantanee de rotație a două repere carteziane triortogonale $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$ ce se mișcă față de reperul fix, în cazul în care cele 2 repere mobile se mișcă exclusiv prin rotație, unul față de celălalt.

Toate reperele carteziane triortogonale considerate sînt alcătuite din vectori unitari.

COUPLED THERMOELASTIC PROBLEM FOR AN INFINITE MEDIUM WITH A CYLINDRICAL CAVITY

BY

ELISABETA RUSU

1. Introduction. The problem of determining the motion produced in an infinite medium by the application of a pressure at wall of a spherical or a cylindrical cavity has been investigated in thermoelasticity by P. Chadwick [1] and R. S. Dhaliwal and M. U. Shanker [2], respectively.

The generalized theory of thermoelasticity has been derived by A. E. Green and K. A. Lindsay [3].

In this paper we consider the corresponding thermoelastic problem in which the cylindrical cavity face is subjected to a step input of stress and zero temperature, in the generalized theory of linear thermoelasticity [4]. The Laplace technique transform is used to develop the solution, for small time.

2. Basic Equations. Let the medium be an infinitely extended, having an infinite cylindrical cavity of radius r_0 . We assume that the medium is at rest and unstrained initially, and the cylindrical cavity face $r=r_0$ is subjected to a step input of stress and zero temperature. (2)

The basic equations of the linear theory of an isotropic solid with a centre of symmetry at each point are [4]:

i) *Equations of motion*

$$t_{ij}^i = \rho \ddot{u}_i, \quad \theta_0 \rho \eta + q_i^i = 0; \quad (3)$$

ii) *Constitutive equations*

$$t_{ij}^i = \lambda e \delta_{ij} + 2\mu \epsilon_{ij}^i - \beta(\theta - \theta_0) \delta_{ij}, \quad (4)$$

$$\rho \eta = a + d\theta + h\dot{\theta} + \beta e, \quad q_i = -\theta_0 k \theta_{,i}; \quad (5)$$

iii) *Geometrical relations*

$$\epsilon_{ij} = (u_{i|j} + u_{j|i}), \quad \epsilon_j^i = g^{ik} \epsilon_{kj}, \quad e = \epsilon_i^i; \quad (6)$$

where u_i — are the components of the displacement vector; θ — the change from the equilibrium temperature θ_0 ; $k, \alpha, d, h, a, \lambda, \mu, \beta$ — characteristic constants of medium.

We denote by (r, θ, z) cylindrical polar coordinates referred to the axis of the cylinder. Let $(\theta^1, \theta^2, \theta^3) = (r, \theta, z)$. The metric tensors and non-zero Christoffel's symbols are

$$g_{11}=g^{11}=g_{33}=g^{33}=1, \quad g_{22}=r^2, \quad g^{22}=r^{-2}, \quad g_{ij}=0 \quad (i \neq j);$$

$$\Gamma_{22}^1 = -r, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}.$$

If $u_1=u(r, t)$, $u_2=0$, $u_3=0$, $\theta=\theta(r, t)$, then

$$\varepsilon_1^1 = \frac{\partial x}{\partial r}, \quad \varepsilon_2^2 = \frac{u}{r}, \quad \varepsilon_3^3 = 0, \quad \varepsilon_j^i = 0 \quad (i \neq j), \quad e = \frac{\partial u}{\partial r} + \frac{u}{r}.$$

The governing equations for a linear theory are then

$$(1) \quad \begin{cases} (\lambda + 2\mu) \left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{u}{r} \right) - \beta \frac{\partial}{\partial r} (\theta + \dot{\alpha} \theta) \right] = \rho \ddot{u}, \\ k \Delta \theta = d \ddot{\theta} + h \ddot{\theta} + \beta \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right). \end{cases}$$

Let us introduce the following non-dimensional quantities defined by

$$(2) \quad \begin{aligned} U &= \frac{\rho a v_1^2}{\beta} u, \quad R = ar, \quad \tau = v_1 at, \quad v_1^2 = \frac{\lambda + 2\mu}{\rho}; \\ A &= \frac{v_1 d}{k}, \quad B = \frac{h v_1^2}{k}, \quad \varepsilon = \frac{\beta^2}{\rho k v_1 a}, \quad m^2 = \frac{\mu}{\lambda + 2\mu} \end{aligned}$$

in Eqs. (1) to obtain

$$(3) \quad \begin{cases} \frac{\partial^2 U}{\partial R^2} + \frac{\partial}{\partial R} \left(\frac{U}{R} \right) - \frac{\partial}{\partial R} \left(\theta + B \frac{\partial \theta}{\partial \tau} \right) - \frac{\partial^2 U}{\partial R^2} = 0, \\ \Delta \theta = \frac{\partial \theta}{\partial \tau} + A \frac{\partial^2 \theta}{\partial \tau^2} + \varepsilon \frac{\partial}{\partial \tau} \left(\frac{\partial U}{\partial R} + \frac{U}{R} \right). \end{cases}$$

We can write initial and regularity conditions, if we assume that the medium is at rest and unstrained initially as follows:

$$(4) \quad U=0, \quad \theta=0; \quad \frac{\partial U}{\partial \tau}=0, \quad \frac{\partial \theta}{\partial \tau}=0 \quad \text{at} \quad \tau=0, \quad R \geq \eta = \frac{d v_1}{a} r_0;$$

$$(5) \quad U=0, \quad \theta=0 \quad \text{for} \quad \tau > 0 \quad \text{when} \quad R \rightarrow \infty.$$

The main aim of our study is to solve the coupled Eqs. (3) subjected to the initial conditions (4) and to the following boundary condition: a constant stress of magnitude T_1 is suddenly applied to the boundary of the cylindrical hole, whereas the temperature at this boundary is kept constant:

$$(6) \quad \begin{cases} \sigma = T_1 H(\tau), \quad t_2^2 = t_3^3 = 0, \quad t_j^i = 0 \quad (i \neq j) \\ \theta = 0 \end{cases} \quad \text{on} \quad R = \eta,$$

where $H(\tau)$ is the Heaviside step function and σ is the dimensionless form of the stress t_1^* which is given as

$$(7) \quad \sigma = \beta \left[\frac{\partial U}{\partial R} + (1 - 2m^2) \frac{U}{R} - \left(\theta + B \frac{\partial \theta}{\partial \tau} \right) \right].$$

3. General Transformed Solution of the Problem. Let „—“ denote the Laplace transform

$$(8) \quad \bar{f}(R, p) = \mathcal{L}(f(R, \tau)) = \int_0^{\infty} f(R, \tau) e^{-p\tau} d\tau.$$

Application of the Laplace transform to Eqs. (3), by using the initial conditions (4), yields

$$(9) \quad (D_3 - p^2) \bar{U} = (1 + Bp) D \bar{\theta}, \quad (D_4 - p - Ap^2) \bar{\theta} = \varepsilon p D_1 \bar{U},$$

where

$$(10) \quad D = \frac{d}{dR}, \quad D_1 = \frac{d}{dR} + \frac{1}{R}, \quad D_3 = DD_1, \quad D_4 = D_1 D.$$

The boundary conditions (6) in the transformed domain are

$$(11) \quad \frac{dU}{dR} + (1 - 2m^2) \frac{U}{R} - (1 + Bp) \bar{\theta} = \frac{T_1}{\beta p}, \quad \bar{\theta} = 0.$$

From (9) we obtain

$$(12) \quad (D_3 - \alpha_i^2) (D_3 - \alpha_j^2) \bar{U} = 0, \quad (D_4 - \alpha_i^2) (D_4 - \alpha_j^2) \bar{\theta} = 0,$$

where α_i^2 ($i=1, 2$) are the roots of the equation

$$(13) \quad \alpha^4 - [p^2 + Ap + p + \varepsilon p (1 + Bp)] \alpha^2 + p^3 (1 + Ap) = 0.$$

The general solution in the transformed domain which is consistent with the regularity conditions (5) is

$$(14) \quad \bar{U} = A_1 K_1(\alpha_1 R) + A_2 K_1(\alpha_2 R), \quad \bar{\theta} = B_1 K_0(\alpha_1 R) + B_2 K_0(\alpha_2 R),$$

where A_1, A_2, B_1, B_2 are constants, K_0 and K_1 are modified Bessel functions.

By substituting (14) in (9) we get

$$(15) \quad B_1 = \frac{p^2 - \alpha_1^2}{\alpha_1 (1 + Bp)} A_1, \quad B_2 = \frac{p^2 - \alpha_2^2}{\alpha_2 (1 + Bp)} A_2.$$

Satisfaction of the boundary conditions (11) yields

$$(16) \quad A_1 = \frac{T_1}{\beta} \frac{(p^2 - \alpha_2^2) \alpha_1 K_0(\alpha_2 \eta)}{p \Delta}, \quad A_2 = - \frac{T_1 (p^2 - \alpha_1^2) \alpha_2 K_0(\alpha_1 \eta)}{\beta p \Delta},$$

$$\Delta = (p^2 - \alpha_1^2) \alpha_2 K_0(\alpha_1 \eta) \left[\alpha_2 K_0(\alpha_2 \eta) + \frac{2m^2}{\eta} K_1(\alpha_2 \eta) \right] -$$

$$- (p^2 - \alpha_2^2) \alpha_1 K_0(\alpha_2 \eta) \left[\alpha_1 K_0(\alpha_1 \eta) + \frac{2m^2}{\eta} K_1(\alpha_1 \eta) \right].$$

4. Short Time Approximation Solution. Let us to obtain inverse Laplace transform, for small values of time. From Abel's theorem it follows that high values of p correspond to small values of τ . Expanding the expressions of roots α_1, α_2 in Mac Laurin series, retaining the first two terms and disregarding ε^2 in comparison with ε , we have [6]

$$(17) \quad \begin{aligned} \alpha_1 &= p \left[1 + \frac{B\varepsilon}{2(1-A)} + \frac{(1-A+B)\varepsilon}{2P(1-A)^2} + \frac{(1-A+B)\varepsilon}{2p^2(1-A)^2} + \dots \right], \\ \alpha_2 &= p\sqrt{A} \left[1 - \frac{B\varepsilon}{2(1-A)} + \frac{1}{2pA} + \frac{(2A^2-AB-2A-B)\varepsilon}{4pA(1-A)^2} - \frac{1}{8A^2p^2} + \right. \\ &\quad \left. + \frac{(4A^3-3A^2B-6AB-4A+B)}{16Ap^2(1-A)^3} + \dots \right]. \end{aligned}$$

By using the asymptotic expansion

$$K_n(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{4n^2-1}{1!(8z)} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!(8z)^2} + \dots \right]$$

we get

$$\begin{aligned} \alpha_1^2 &= p^2 \left[1 + \frac{B\varepsilon}{1-A} + \frac{(1-A+B)\varepsilon}{p(1-A)^2} + \frac{(1-A+B)\varepsilon}{p^2(1-A)^3} + \dots \right], \\ \alpha_2^2 &= p^2 \left[A - \frac{AB\varepsilon}{1-A} + \frac{1}{p} + \frac{(A^2-A-B)\varepsilon}{p(1-A)^2} - \frac{(1-A+B)\varepsilon}{p^2(1-A)^3} + \dots \right], \\ \alpha_1^2 \alpha_2^2 &= Ap^4 \left(1 + \frac{1}{Ap} \right); \\ \frac{1}{\Delta} &= -\frac{\frac{2\eta}{\pi} \sqrt{\alpha_1 \alpha_2} e^{\alpha_1 \eta} e^{\alpha_2 \eta}}{p^4(1-A)} - \left\{ 1 - \frac{B(1+A)\varepsilon}{(1-A)^2} + \frac{1}{p} \left[\frac{1}{1-A} - \frac{(1+2B-A^2)\varepsilon}{(1-A)^3} - \right. \right. \\ &\quad \left. \left. - \frac{2B\varepsilon(1+A)}{(1-A)^3} \right] + \frac{1}{8\eta p} \left[1 + \frac{1}{\sqrt{A}} - \frac{B\varepsilon}{2(1-A)} + \frac{B\varepsilon}{2\sqrt{A}(1-A)} - \right. \right. \\ &\quad \left. \left. - \frac{B\varepsilon(1+A)(A+1)}{\sqrt{A}(1-A)^2} \right] - \frac{2m^2}{\eta p} \left[1 + \frac{\varepsilon \sqrt{A} B}{(1-A)^2} + \frac{B\varepsilon(1+A)}{2(1-A)^2} - \frac{2B\varepsilon(1+A)}{(1-A)^2} \right] \right\}, \\ A_1 &= -\frac{T_1}{\beta} \sqrt{\frac{\pi}{2\eta \alpha_2}} \frac{2\eta}{\pi} \sqrt{\alpha_1 \alpha_2} \frac{1}{p^2} \left\{ 1 + \frac{1}{p} \left[-\frac{(1+2B-A^2)\varepsilon}{2(1-A)^3} - \frac{B\varepsilon(1+A)}{2(1-A)^3} \right] - \right. \\ &\quad \left. - \frac{2m^2}{\eta p} \left[1 + \frac{\varepsilon \sqrt{A} B}{(1-A)^2} - \frac{B\varepsilon(1+A)}{(1-A)^2} \right] + \frac{1}{8\eta p} \left[1 - \frac{B\varepsilon}{(1-A)^2} \right] \right\}, \\ A_2 &= -\frac{T_1}{\beta} \sqrt{\frac{\pi}{2\alpha_1 \eta}} \frac{2\eta}{\pi} \sqrt{\alpha_1 \alpha_2} e^{\alpha_1 \eta} \frac{\varepsilon \sqrt{A}}{p^2(1-A)} \left\{ \frac{B}{1-A} + \frac{1}{p} \left[\frac{1-A+2B}{(1-A)^2} + \right. \right. \\ &\quad \left. \left. + \frac{B}{2A(1-A)} \right] + \frac{1}{8\eta p} \frac{B}{\sqrt{A}(1-A)} - \frac{2m^2}{\eta p} \frac{B}{1-A} \right\}; \end{aligned}$$

$$A_1 K_1(\alpha_1 R) = -\frac{T_1}{\beta} \sqrt{\frac{\eta}{R}} e^{-\alpha_1(R-\eta)} \frac{1}{p^2} \left\{ 1 + \frac{1}{p} \left[-\frac{(1+2B-A^2)\varepsilon}{2(1-A)^3} - \frac{B\varepsilon(1+A)}{2(1-A)^3} \right] + \right. \\ \left. + \frac{1}{8\eta p} \left[1 - \frac{B\varepsilon}{(1-A)^2} \right] - \frac{2m^2}{\eta p} \left[1 + \frac{\varepsilon\sqrt{A}B}{(1-A)^2} - \frac{B\varepsilon(1+A)}{(1-A)^2} \right] + \right. \\ \left. + \frac{3}{8Rp} \left[1 - \frac{B\varepsilon}{2(1-A)} \right] \right\},$$

$$A_2 K_1(\alpha_2 R) = -\frac{T_1}{\beta} \sqrt{\frac{\eta}{R}} e^{-\alpha_2(R-\eta)} \frac{\varepsilon\sqrt{A}}{p^2(1-A)} \left\{ \frac{B}{1-A} + \frac{1}{p} \left[\frac{1-A+2B}{(1-A)^2} + \right. \right. \\ \left. \left. + \frac{B}{2A(1-A)} \right] + \frac{1}{8\eta p} \frac{B}{\sqrt{A}(1-A)} + \frac{3}{8R\sqrt{A}p} \frac{B}{1-A} - \frac{2m^2}{\eta p} \frac{B}{1-A} \right\}.$$

From (14) we find

$$\bar{U}(R, p) = -\frac{T_1}{\beta} \sqrt{\frac{\eta}{R}} \left\{ \frac{1}{p^2} e^{-\alpha_1(R-\eta)} \left\{ 1 + \frac{1}{p} \left[-\frac{(1+2B-A^2)\varepsilon}{2(1-A)^3} - \frac{B\varepsilon(1+A)}{2(1-A)^3} \right] + \right. \right. \\ \left. \left. + \frac{1}{8\eta p} \left[1 - \frac{B\varepsilon}{(1-A)^2} \right] - \frac{2m^2}{\eta p} \left[1 + \frac{\varepsilon\sqrt{A}B}{(1-A)^2} - \frac{B\varepsilon(1+A)}{(1-A)^2} \right] + \right. \right. \\ \left. \left. + \frac{3}{8Rp} \left[1 - \frac{B\varepsilon}{2(1-A)} \right] \right\} + \frac{\varepsilon\sqrt{A}}{p^2(1-A)} e^{-\alpha_2(R-\eta)} \left\{ \frac{B}{1-A} + \right. \right. \\ \left. \left. + \frac{1}{p} \left[\frac{1-A+2B}{(1-A)^2} + \frac{B}{2A(1-A)} \right] + \frac{B}{8\eta p\sqrt{A}(1-A)} + \right. \right. \\ \left. \left. + \frac{3B}{8Rp\sqrt{A}(1-A)} - \frac{2m^2 B}{\eta p(1-A)} \right\} \right\},$$

$$\bar{\theta}(R, p) = \frac{T_1\varepsilon}{\beta} \sqrt{\frac{\eta}{R}} \left\{ \frac{1}{(1-A)p^2} e^{-\alpha_1(R-\eta)} \left[1 + \frac{1}{p(1-A)} + \frac{1}{8\eta p} - \frac{1}{8Rp} - \frac{2m^2}{\eta p} \right] - \right. \\ \left. - \frac{1}{(1-A)p^2} e^{-\alpha_2(R-\eta)} \left[1 + \frac{1}{p(1-A)} + \frac{1}{8\eta p\sqrt{A}} - \frac{1}{8Rp\sqrt{A}} - \frac{2m^2}{\eta p} \right] \right\}.$$

We denote by

$$\lambda_1 = (R-\eta) \left[1 + \frac{B\varepsilon}{2(1-A)} \right], \quad \omega_1 = (R-\eta) \frac{1-A+B}{(1-A)^3}, \quad \omega_2 = (R-\eta) \frac{1}{8A\sqrt{A}},$$

$$\lambda_2 = \frac{R-\eta}{2\sqrt{A}} \left[1 + \frac{(A^2-AB-2A-B)\varepsilon}{2(1-A)^2} \right],$$

$$C = (R-\eta) \frac{4A^3-3A^2B-6AB-4A+B}{16A\sqrt{A}(1-A)^3}.$$

Then

$$\begin{aligned}\bar{U}(R, p) = & -\frac{T_1}{\beta} \sqrt{\frac{\eta}{R}} \left\{ \frac{1}{p^2} e^{-\lambda_1 p} e^{-(1-A)\omega_1 \varepsilon} \left\{ 1 + \frac{1}{p} \left[-\frac{(1+2B-A^2)\varepsilon}{2(1-A)^3} - \right. \right. \right. \\ & \left. \left. - \frac{B\varepsilon(1+A)}{2(1-A)^2} \right] + \frac{1}{8\eta p} \left[1 - \frac{B\varepsilon}{(1-A)^2} \right] - \frac{2m^2}{\eta p} \left[1 + \frac{\varepsilon\sqrt{A}B}{(1-A)^2} - \frac{B\varepsilon(1+A)}{(1-A)^2} \right] + \right. \\ & \left. + \frac{3}{8R p} \left[1 - \frac{B\varepsilon}{2(1-A)} \right] - \frac{\omega_1 \varepsilon}{p} \right\} + \frac{\varepsilon\sqrt{A}}{p^2(1-A)} e^{-\lambda_2 p} e^{-\lambda_3} \left\{ \frac{B}{1-A} + \frac{1}{p} \left[\frac{1-A+2B}{(1-A)^2} + \right. \right. \\ & \left. \left. + \frac{B}{2A(1-A)} \right] + \frac{B}{8\eta p\sqrt{A}(1-A)} + \frac{3B}{8R p\sqrt{A}(1-A)} - \frac{2m^2 B}{\eta p(1-A)} + \frac{\omega_2 B}{p(1-A)} \right\} \Bigg\}, \\ \bar{\theta}(R, p) = & \frac{T_1 \varepsilon}{\beta} \sqrt{\frac{\eta}{R}} \left\{ \frac{1}{(1-A)p^2} e^{-\lambda_1 p} e^{-(1-A)\omega_1 \varepsilon} \left[1 + \frac{1}{p(1-A)} + \frac{1}{8\eta p} - \right. \right. \\ & \left. \left. - \frac{1}{8R p} - \frac{2m^2}{\eta p} \right] - \frac{1}{(1-A)p^2} e^{-\lambda_2 p} e^{-\lambda_3} \left[1 + \frac{1}{p(1-A)} + \frac{1}{8\eta p\sqrt{A}} - \right. \right. \\ & \left. \left. - \frac{1}{8R p\sqrt{A}} - \frac{2m^2}{p} - \frac{\omega_2}{p} \right] \right\}.\end{aligned}$$

In [3] one proves that $\alpha d - h \geq 0$. It follows that $1-A+B \geq 0$ and $(1-A)\omega_1 \varepsilon \geq 0$ for $R \geq \eta$ and because h and ε are small, $\lambda_2 \geq 0$ and $\lambda_3 \geq 0$.

We perform the inverse Laplace transform and we find the solution of boundary-value problem, for small time

$$\begin{aligned}U(R, \tau) = & -\frac{T_1}{\beta} \sqrt{\frac{\eta}{R}} \left\{ e^{-(1-A)\omega_1 \varepsilon} \left\{ H(\tau-\lambda_1)(\tau-\lambda_1) + \frac{1}{2} H(\tau-\lambda_1)(\tau-\lambda_1)^2 \times \right. \right. \\ & \times \left\{ -\frac{(1+2B-A^2)\varepsilon}{2(1-A)^3} - \frac{B\varepsilon(1+A)}{2(1-A)^2} + \frac{1}{8\eta} \left[1 - \frac{B\varepsilon}{(1-A)^2} \right] - \right. \\ & \left. \left. - \frac{2m^2}{\eta} \left[1 + \frac{\varepsilon\sqrt{A}B}{(1-A)^2} - \frac{B\varepsilon(1+A)}{(1-A)^2} \right] + \frac{3}{8R} \left[1 - \frac{B\varepsilon}{2(1-A)} \right] - \omega_1 \varepsilon \right\} \right\} + \\ & + \frac{\varepsilon\sqrt{A}}{1-A} e^{-\lambda_3} \left\{ \frac{B}{1-A} H(\tau-\lambda_2)(\tau-\lambda_2) + \frac{1}{2} H(\tau-\lambda_2)(\tau-\lambda_2)^2 \left[\frac{1-A+2B}{(1-A)^2} + \right. \right. \\ & \left. \left. + \frac{B}{2A(1-A)} + \frac{B}{8\eta\sqrt{A}(1-A)} + \frac{3B}{8R\sqrt{A}(1-A)} - \frac{2m^2 B}{\eta(1-A)} + \frac{\omega_2 B}{1-A} \right] \right\} \Bigg\}, \\ \theta(R, \tau) = & \frac{T_1 \varepsilon}{\beta} \sqrt{\frac{\eta}{R}} \left\{ \frac{1}{1-A} e^{-(1-A)\omega_1 \varepsilon} \left[H(\tau-\lambda_1)(\tau-\lambda_1) + \frac{1}{2} H(\tau-\lambda_1)(\tau-\lambda_1)^2 \times \right. \right. \\ & \times \left(\frac{1}{1-A} + \frac{1}{8\eta} - \frac{1}{8R} - \frac{2m^2}{\eta} \right) \Bigg] - \frac{1}{1-A} e^{-\lambda_3} \left[H(\tau-\lambda_2)(\tau-\lambda_2) + \right. \\ & \left. \left. + \frac{1}{2} H(\tau-\lambda_2)(\tau-\lambda_2)^2 \left(\frac{1}{1-A} + \frac{1}{8\eta\sqrt{A}} - \frac{1}{8R\sqrt{A}} - \frac{2m^2}{\eta} - \omega_2 \right) \right] \right\}.\end{aligned}$$

Note that in the generalized theory of linear thermoelasticity U and θ are continuous and there are not displacement and temperature at any R prior to arrival of the wave; there are two fronts of wave which arrive in any R .

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REFERENCES

1. Chadwick P., *Thermoelasticity. The Dynamical Theory*. Progress in Solid Mech., Vol. I, Ed. Sneddon and Hill, 1960.
2. Dhaliwal R. S., Shanker M. U., *Coupled Thermoelastic Problem for an Infinite Medium with a Cylindrical Hole*. Arch. Mech. Stos., **XXII**, 5, 531–546 (1970).
3. Green A. E., Lindsay K. A., *Thermoelasticity*. J. Elasticity, 2, 1–7 (1972).
4. Green A. E., *A Note on Linear Thermoelasticity*. Mathematica, 19, Part I, 37, 69–76 (1972).
5. Rusu E., *Asupra unei probleme de dinamică pentru mediul infinit cu gol cilindric în teoria generalizată a termoeleasticității*. Ses. șt. „Creația tehnică și fiabilitatea în construcția de mașini”, Iași, noiembrie 1985.
6. Nistor I., *Fundamental Solution in a Generalized Theory of Thermoelasticity for Small Time*. Proc. Inst. Math. Iași, Acad. R. S. R., 287–295 (1975).

PROBLEMĂ DE TERMOELASTICITATE CUPLATĂ PENTRU UN MEDIU ÎNFINIT CU GOL CILINDRIC

(Rezumat)

Se determină deplasarea și temperatura într-un mediu infinit cu gol cilindric, pe frontiera căruia se aplică brusc o tensiune constantă.

Note that in the generalized theory of linear thermodynamics (1) and (2) are continuous and there are no discontinuities and temperature at any R prior to arrival of the wave; there are two kinds of wave which arrive in any R .

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REFERENCES

1. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).
2. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).
3. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).
4. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).
5. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).
6. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).
7. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).
8. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).
9. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).
10. E. G. Lusk, *J. Appl. Phys.*, **27**, 1000 (1956).

PROCEEDINGS OF THE AMERICAN PHYSICAL SOCIETY
AT THE UNIVERSITY OF CALIFORNIA, SAN DIEGO

January 1, 1956

25. Theoretical aspects of the theory of the electron in a magnetic field. The theory of the electron in a magnetic field is a topic of great interest in the theory of the electron in a magnetic field.

THERMAL SHOCK ON THE BOUNDARY OF ELASTIC MICROPOLAR HALF SPACE

BY

SILVIA-GABRIELA CIUMAȘU

1. Introduction. The problem of propagation of thermal stresses in an elastic half space, arising after a sudden heating of its boundary was considered by V.I. D a n i l o v s k a i a [2] in the linear theory of uncoupled thermoelasticity.

In this paper a problem of propagation of thermal stresses will be considered in the context of generalized theory of micropolar uncoupled thermoelasticity. For homogeneous micropolar continua the generalized theory of thermoelasticity has been derived by E. B o s c h i and D. I e ș a n [1]. Because difficulty to calculate of inverse Laplace transforms usually is used to develop the solutions for small values of time [3], [5], [6]. In our problem, the method of Laplace transforms is used to develop the solutions and the inverse Laplace transforms can be directly obtained.

2. Basic Equations. We refer the motion of continuum to a fixed system of rectangular cartesian axes. For a homogeneous, isotropic and centrosymmetric body occupying the region V , with the boundary ∂V , the basic equations in the linear generalized theory of micropolar thermoelasticity are [1]:

i) *Equations of motions*

$$(2.1) \quad \begin{aligned} (\lambda + \mu) u_{j,j} + (\mu + \kappa) u_{i,jj} + \kappa \varepsilon_{ijk} \varphi_{k,j} - m(\theta + \alpha \dot{\theta})_{,i} + \rho f_i &= \rho \ddot{u}_i, \\ (\alpha_1 + \beta) \varphi_{j,j} + \gamma \varphi_{i,jj} + \kappa \varepsilon_{ijk} u_{k,j} - 2\kappa \varphi_i + \rho l_i &= I \ddot{\varphi}_i. \end{aligned}$$

ii) *Equations of energy*

$$(2.2) \quad \theta_0 \rho \dot{S} + q_{i,i} = \rho r.$$

iii) *Constitutive Equations*

$$(2.3) \quad \begin{aligned} t_{ij} &= \lambda e_{rr} \delta_{ij} + (\mu + \kappa) e_{ij} + \mu e_{ji} - m(\theta + \alpha \dot{\theta}) \delta_{ij}, \\ m_{ji} &= \alpha_1 \kappa_{rr} \delta_{ij} + \beta \kappa_{ji} + \gamma \kappa_{ij}, \\ q_i &= -\theta_0 k \theta_{,i}, \\ \rho S &= a + d\theta + h\dot{\theta} + m e_{jj}. \end{aligned}$$

iv) Kinematic relations

$$(2.4) \quad e_{ij} = u_{j,i} + \varepsilon_{ijk} \varphi_k, \quad \chi_{ij} = \varphi_{j,i}.$$

From the residual entropy inequality we have the condition [1]

$$(\alpha d - h) \dot{\theta}^2 + k \dot{\theta}, \dot{\theta} \geq 0.$$

In the above equations we have used the following notations: u_i , φ_i — components of the displacements vector and rotation vector, respectively; e_{ij} , χ_{ij} — kinematic characteristics of the strain; t_{ij} , m_{ij} — components of force stress and moment stress tensors; S — specific entropy; ρ — density; r — externally applied heat per unit mass; q — flux of heat vector; I — rotational inertia; f_i , l_i — components of body forces and the body couples respectively; θ_0 — constant reference temperature; λ , μ , κ , γ , α_1 , m , α , k , d , h — characteristic constants of the material.

3. A Problem of Propagation of Thermal Stresses. Let V be from now on the open half space ($-\infty < x_2, x_3 < \infty$; $0 < x_1 < \infty$) so that ∂V is the plane $x_1 = 0$. We suppose that the body forces, the body couples and the externally supplied specific heat are absent.

The initial conditions are

$$(3.1) \quad \begin{aligned} u_i(x_1, 0) &= 0, & \varphi_i(x_1, 0) &= 0, & \theta(x_1, 0) &= 0, & \frac{\partial \theta}{\partial x_1}(x_1, 0) &= 0; \\ \dot{u}_i(x_1, 0) &= 0, & \dot{\varphi}_i(x_1, 0) &= 0, & \dot{\theta}(x_1, 0) &= 0 & (x_1 \geq 0, \tau = 0); \end{aligned}$$

on the $x_1 = 0$ we have the boundary conditions

$$(3.2) \quad u_i(0, \tau) = 0, \quad \varphi_i(0, \tau) = 0, \quad \frac{\partial \theta(0, \tau)}{\partial x_1} = T^* H(\tau) \quad (x_1 = 0, \tau > 0),$$

where $H(\tau)$ is the Heaviside function.

In this problem we have

$$(3.3) \quad u_i = u_i(x_1, t), \quad \varphi_i = \varphi_i(x_1, t), \quad \theta = \theta(x_1, t).$$

From (2.1), (2.2), (3.3) we obtain

$$(3.4) \quad \begin{aligned} (\lambda + 2\mu + \kappa) u_{1,11} - m(\theta + \alpha \dot{\theta}) &= \rho \ddot{u}_1, \\ (\mu + \kappa) u_{2,11} - \kappa \varphi_{3,1} - \rho \ddot{u}_2 &= 0, \\ (\mu + \kappa) u_{3,11} + \kappa \varphi_{2,1} - \rho \ddot{u}_3 &= 0, \\ (\alpha + \beta + \gamma) \varphi_{1,11} - 2\kappa \varphi_1 - I \ddot{\varphi}_1 &= 0, \\ \gamma \varphi_{2,11} - \kappa u_{3,1} - 2\kappa \varphi_2 - I \ddot{\varphi}_2 &= 0, \\ \gamma \varphi_{3,11} + \kappa u_{2,1} - 2\kappa \varphi_3 - I \ddot{\varphi}_3 &= 0, \\ k \theta_{,11} - d \dot{\theta} - h \ddot{\theta} - m \dot{u}_{1,1} &= 0. \end{aligned}$$

It is convenient to write the basic Eqs. (3.4) in dimensionless form. We consider the following variables

$$(3.5) \quad \tau = w^* t, \quad z = \frac{w^*}{v_p} x_1, \quad u = \frac{w^*}{v_p} x_1, \quad T = \frac{\theta}{\theta_0},$$

$$v_p = \sqrt{\frac{\lambda + 2\mu + \kappa}{\rho}}, \quad w^* = \frac{v_p^2 d}{k}.$$

The Eqs. (3.1)₁, (3.1)₇ in which the temperature θ is coupled with the displacement component u_1 , respectively assume the form

$$(3.6) \quad \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2} \right) u - \frac{\nu}{\beta^2} \frac{\partial}{\partial z} (T + \alpha \dot{T}) = 0,$$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial \tau} \right) T - c \frac{\partial^2 T}{\partial \tau^2} - g \frac{\partial^2 u}{\partial z \partial \tau} = 0,$$

where $\beta^2 = \rho v_p^2$, $c = h v_p^2$, $g = m / (\theta_0 d)$, $\nu = \theta_0 m$.

In uncoupled thermoelastic problem we have the equations

$$(3.7) \quad \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2} \right) u - \frac{\nu}{\beta^2} \frac{\partial}{\partial z} (T + \alpha \dot{T}) = 0,$$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial \tau} \right) T - c \frac{\partial^2 T}{\partial \tau^2} = 0.$$

Let „—“ denote the Laplace transform of function, defined by

$$\bar{f}(z, p) = \mathcal{L}[f(z, \tau)] = \int_0^\infty e^{-p\tau} f(z, \tau) d\tau.$$

Performing in (3.6) the Laplace transform we obtain

$$(3.8) \quad \left(\frac{d^2}{dz^2} - p^2 \right) \bar{u} - \frac{\nu}{\beta^2} \frac{d\bar{T}}{dz} (1 + \alpha p) = 0,$$

$$\frac{d^2 \bar{T}}{dz^2} - (p + p^2 c) \bar{T} = 0.$$

From the boundary conditions we obtain

$$(3.9) \quad \bar{u}(0, p) = 0, \quad \frac{dT(0, p)}{dz} = \frac{v_p \bar{T}^*}{\theta_0 w^*} \frac{1}{p}.$$

On the assumption that u and θ tend to zero as $x \rightarrow \infty$, we have solutions of Eqs. (3.8), (3.9) of the form

$$(3.10) \quad \begin{aligned} \bar{T} &= -\frac{v_p T^*}{\theta_0 w^*} \frac{e^{-z\sqrt{p+p^2c}}}{p\sqrt{p+p^2c}}, \\ \bar{u} &= \frac{v_p T^*}{\beta^2 \theta_0 w^*} \frac{1+\alpha p}{p^2[p(c-1)+1]} [e^{-z\sqrt{p+p^2c}} - e^{-pz}]. \end{aligned}$$

From (2.3), (3.3), (3.5), we have

$$(3.11) \quad \bar{t}_{11} = \beta^2 \frac{d\bar{u}}{dz} - v\bar{T}(1+\alpha p)$$

(analogous we obtain for $\bar{t}_{22} = \bar{t}_{33}$).

Using (3.10), (3.11) we have

$$(3.12) \quad \begin{aligned} \bar{t}_{11} &= \frac{v T^* v_p}{\theta_0 w^*} \left\{ \frac{1+\alpha p}{p^2[p(c-1)+1]} \left[-\sqrt{p+p^2c} e^{-z\sqrt{p+p^2c}} + p e^{-pz} \right] + \right. \\ &\quad \left. + \frac{e^{-z\sqrt{p+p^2c}}}{p\sqrt{p+p^2c}} (1+\alpha p) \right\}. \end{aligned}$$

We obtain the inverse Laplace transforms (3.10), (3.12) using Borel's theorem [4] and the relations [7]

$$(3.13) \quad \begin{aligned} \mathcal{L}^{-1}\left(\frac{e^{-pz}}{p}\right) &= H(\tau-z), \quad z > 0, \quad \operatorname{Re} p > 0; \\ \mathcal{L}^{-1}\left(\frac{1}{p-a}\right) &= e^{a\tau}, \quad \mathcal{L}^{-1}\left(\frac{1}{p}\right) = 1, \quad \mathcal{L}^{-1}\left(\frac{1}{p^2}\right) = \tau, \quad \tau > 0; \\ \mathcal{L}^{-1}\left(\frac{e^{-b\sqrt{p^2-a^2}}}{\sqrt{p^2-a^2}}\right) &= H(\tau-b) I_0[a\sqrt{\tau^2-b^2}], \quad b > 0, \quad \operatorname{Re} p > \operatorname{Re} a; \\ \mathcal{L}^{-1}\left(\frac{e^{-b(p-a)}}{p-a}\right) &= e^{a\tau} H(\tau-b), \quad b > 0, \quad \operatorname{Re} p > \operatorname{Re} a; \\ \mathcal{L}^{-1}\left(e^{b(p-\sqrt{p^2-a^2})}-1\right) &= \frac{ab I_1[a(\tau^2+2b\tau)^{1/2}]}{\sqrt{\tau^2+2b\tau}}, \quad \operatorname{Re} p > \operatorname{Re} a; \\ \mathcal{L}^{-1}\left(\frac{pe^{-b\sqrt{p^2-a^2}}}{\sqrt{p^2-a^2}} - e^{-bp}\right) &= H(\tau-b) \frac{a\tau I_1(a\sqrt{\tau^2-b^2})}{\sqrt{\tau^2-b^2}}, \quad b > 0, \end{aligned}$$

where I_0, I_1 are the modified Bessel function.

We find the solution of boundary-value problem (3.4), (3.1), (3.2)

$$\begin{aligned} T(z, \tau) &= -\frac{T^* v_p}{w^* \theta_0 \sqrt{c}} H(\tau - \sqrt{cz}) F_1(z, \tau), \\ u_1(z, \tau) &= \frac{T^* v_p}{w^* \beta^2 \theta_0} [-H(\tau-z) F_2(z, \tau) + H(\tau - \sqrt{cz}) F_3(z, \tau)], \end{aligned}$$

$$\begin{aligned}
 u_2(x_1, t) &= u_3(x_1, t) = 0, & \varphi_i(x_1, t) &= 0, \\
 t_{11}(z, \tau) &= \frac{\nu T^* \nu_p}{\theta_0 w^*} \{ H(\tau - \sqrt{cz}) [F_4(z, \tau) + F_5(z, \tau) + F_6(z, \tau) + \\
 &+ F_7(z, \tau) + F_8(z, \tau)] + H(\tau - z) F_9(z, \tau) \}, \\
 m_{ij}(x_1, t) &= 0,
 \end{aligned}$$

where

$$\begin{aligned}
 F_1(z, \tau) &= \int_{\sqrt{cz}}^{\tau} e^{-\frac{\nu}{2c}} I_0 \left[\frac{1}{2c} (v^2 - cz^2)^{1/2} \right] dv, \\
 F_2(z, \tau) &= \int_z^{\tau} \left[1 + \frac{\alpha + 1 - c}{c - 1} e^{-\frac{1}{c-1}(\tau-v)} \right] dv, \\
 F_3(z, \tau) &= \int_{\sqrt{cz}}^{\tau} e^{-\frac{z}{2\sqrt{c}}} + \frac{z}{\sqrt{c}} \int_{\sqrt{cz}}^{\tau} e^{-\frac{t}{2c}} I_1 \left[\frac{1}{2c} \sqrt{t^2 - cz^2} \frac{dt}{\sqrt{t^2 - cz^2}} \right] \times \\
 &\quad \times \left[1 + \frac{\alpha + 1 - c}{c - 1} e^{-\frac{1}{c-1}(\tau-v)} \right] dv, \\
 F_4(z, \tau) &= -\frac{1}{2c\sqrt{c}} \int_{\sqrt{cz}}^{\tau} e^{-\frac{\nu}{2c}} \left[A + B(\tau - v) + \frac{D}{c-1} e^{-\frac{\tau-v}{c-1}} \right] I_0 \left[\frac{1}{2c} \sqrt{v^2 - cz^2} \right] dv, \\
 F_5(z, \tau) &= \frac{1}{2c\sqrt{c}} \int_{\sqrt{cz}}^{\tau} e^{-\frac{\nu}{2c}} \left[A + B(\tau - v) + \frac{D e^{-\frac{\tau-v}{c-1}}}{c-1} \right] \frac{v I_1 \left(\frac{1}{2c} \sqrt{v^2 - cz^2} \right)}{\sqrt{v^2 - cz^2}} dv, \\
 F_6(z, \tau) &= e^{-\frac{z}{2\sqrt{c}}} \frac{1}{\sqrt{c}} \left[A + B(\tau - \sqrt{cz}) + \frac{D}{c-1} e^{-\frac{1}{c-1}(\tau - \sqrt{cz})} \right], \\
 F_7(z, \tau) &= \int_{\sqrt{cz}}^{\tau} e^{-\frac{\nu}{2c}} I_0 \left[\frac{1}{2c} (v^2 - cz^2)^{1/2} \right] dv, \\
 F_8(z, \tau) &= \frac{\alpha}{\sqrt{c}} e^{-\frac{\tau}{2c}} I_0 \left[\frac{1}{2c} (\tau^2 - cz^2)^{1/2} \right], \\
 F_9(z, \tau) &= 1 + \frac{\alpha + 1 - c}{c - 1} e^{-\frac{\tau-z}{c-1}},
 \end{aligned}$$

$$A = \frac{a(c+\alpha)-1}{a^2(c-1)}, \quad B = \frac{(c+\alpha)[a(c-1)-1]-3}{a(c-1)^2}, \quad D = \frac{(1-\alpha a)(1-ca)}{a^2}.$$

We note that in the generalized theory of linear micropolar thermoelasticity u_1 and θ are continuous on the front elastic waves and on the front thermal waves

$$[\theta(z, \tau)]_{\tau=\sqrt{cz}}=0, \quad [u(z, \tau)]_{\tau=z}=0, \quad [u(z, \tau)]_{\tau=\sqrt{cz}}=0.$$

There are two fronts of wave which arrive in any x_1 . The front elastic wave move along the x_1 -axis with the velocity v_p , and the front thermal wave move along the x_1 -axis with the velocity $1/\sqrt{h}$.

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REFERENCES

1. Boschi E., Ieșan D., *A Generalized Theory of Linear Micropolar Thermoelasticity*. Mecanica, VIII, 3 (1973).
2. Danilovskaia V. I., *Thermal Stresses in an Elastic Half Space Arising after a Sudden Heating of its Boundary* (in Russian). P.M.M., 4, 316–318 (1950).
3. Hetnarski R. B., *Coupled Thermoelastic Problem for Half Space*. Bull. Acad. Polon. Sci., Sér. Sci. Techn., 12, 49–58 (1964).
4. Iacob C., Cocirlan P., Dragoș L. Gheorghiiță Șt., Homentcovich D., Nicolau A., Soos E., *Matematici clasice și moderne*. Ed. tehn., Buc., 1981.
5. Lessen M., *Thermoelasticity and Thermal Shock*. J. Mech. Phys. Solids, 5, 57–60 (1956).
6. Rusu E., Crăciunaș A., *Un problème de dynamique pour le demispace dans la théorie généralisée de la thermoélasticité linéaire*. Bul. Inst. polit. Iași, XXVI (XXX), 3–4, s. IV (1980).

ȘOC TERMIC PE FRONTIERA UNUI SEMISPAȚIU MICROPOLAR ELASTIC

(Rezumat)

Se studiază o problemă referitoare la propagarea undelor neperiodice în contextul teoriei generalizate a termoelasticității micropolare necuplate. Metoda de rezolvare este a aplicării transformatorilor Laplace ecuațiilor obținute, inversele transformatorilor putînd fi găsite direct. Rezultatele obținute pun în evidență propagarea a două fronturi de undă, elastic și termic cu viteză de propagare finită.

A RECIPROCITY THEOREM AND A VARIATIONAL THEOREM FOR ELASTIC MEDIA WITH MICROSTRUCTURE

BY

D. MANOLE

0. Introduction. A reciprocity theorem for elastic media with microstructure which does not use the Laplace transform is given [2]. Reciprocity theorems for elastic and thermoelastic homogeneous and isotropic media which use the Laplace transform are given in [4]. We shall now present a reciprocity theorem for elastic media with microstructure which uses the Laplace transform, as well as a variational theorem which follows the method showed in [1].

1. Basic Equations. Consider a elastic medium with microstructure occupying the Ω domain of the three-dimensional Euclidian space with closure $\bar{\Omega}$ and (regular) boundary Σ . The basic equations of linear elasticity theory microstructure are [1]

i) *Equations of motion*

$$(1.1) \quad (\tau_{ji} + \sigma_{ij})_{,j} + \rho f_i = \rho \ddot{u}_i, \\ \mu_{kij,k} + \sigma_{ij} + \rho l_{ij} = I_{is} \ddot{\psi}_{sj}, \quad \tau_{ij} = \tau_{ji}.$$

ii) *Geometrical equations*

$$(1.2) \quad \varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}), \quad \gamma_{ij} = u_{j,i} - \psi_{ij}, \quad \chi_{ijk} = \psi_{jk,i}.$$

iii) *Constitutive equations*

$$(1.3) \quad \tau_{ij} = C_{ijkl} \varepsilon_{kl} + G_{kl ij} \gamma_{kl} + F_{pqrij} \chi_{pqr}, \\ \sigma_{ij} = G_{ijkl} \varepsilon_{kl} + B_{kl ij} \gamma_{kl} + D_{ij pqr} \chi_{pqr}, \\ \mu_{ijk} = F_{ijkrs} \varepsilon_{rs} + D_{rsijk} \gamma_{rs} + A_{ijkpqr} \chi_{pqr}, \quad \text{on } \Omega \times (0, t).$$

The new notations are u_i — components of the displacement vector; ψ_{ij} — components of micro-displacement tensor, ε_{ij} , γ_{ij} , χ_{ijk} — kinematic characteristics of the strain; f_i — components of body force; l_{ij} — components of body microforce; τ_{ij} — components of the classical stress tensor; σ_{ij} — components of the relative stress tensor; μ_{ijk} — components of the couple-

stress tensor; $\rho(x)$ the mass density, $I_{ij}(x)$, $C_{ijkl}(x)$, $G_{ijkl}(x)$, $B_{ijkl}(x)$, $D_{ijkl}(x)$, $F_{pqrij}(x)$, $A_{ijkpqr}(x)$ — characteristic functions of the material; the comma donotes partial derivation with respect to the space variables x_i and a dot denotes partial derivation with respect to the time t .

The characteristic functions of the material obey the simetry relations

$$(1.4) \quad \begin{aligned} I_{ij} &= I_{ji}, & C_{ijkl} &= C_{klij} = C_{jikl}, & B_{ijkl} &= B_{klij} \\ A_{ijkpqr} &= A_{pqrijk}, & F_{ijkrs} &= F_{ijksr}, & G_{ijkl} &= G_{ijlk}. \end{aligned}$$

To system of field equations we adjoin the initial conditions

$$(1.5) \quad u_i(x, 0) = a_i, \quad \dot{u}_i(x, 0) = b_i, \quad \psi_{ij}(x, 0) = c_{ij}, \quad \dot{\psi}_{ij}(x, 0) = d_{ij}$$

and boundary conditions

$$(1.6) \quad \begin{aligned} u_i &= \tilde{u}_i \text{ on } \Sigma_1 \times [0, t_1], & t_i &= (\tau_{ji} + \sigma_{ji}) n_j = \tilde{t}_i \text{ on } \Sigma_2 \times [0, t_1], \\ \psi_{ij} &= \tilde{\psi}_{ij} \text{ on } \Sigma_3 \times [0, t_1], & T_{ij} &= \mu_{kij} n_k = \tilde{T}_{ij} \text{ on } \Sigma_4 \times [0, t_1], \end{aligned}$$

where a_i , b_i , c_{ij} , d_{ij} , $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\psi}_{ij}$, $\tilde{T}_{ij}$ are prescribed functions, n_i are components of the unit outward normal to Σ and Σ_γ ($\gamma=1, 2, 3, 4$) are parts of Σ such that $\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \Sigma$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$,

2. Reciprocity Theorem. *If an elastic solid with microstructure is subjected to two different systems of elastic loadings*

$$L^{(\alpha)} = \{f_i^{(\alpha)}, l_{ij}^{(\alpha)}, \tilde{u}_i^{(\alpha)}, \tilde{\psi}_{ij}^{(\alpha)}, \tilde{t}_i^{(\alpha)}, \tilde{T}_{ij}^{(\alpha)}\}, \quad (\alpha=1, 2),$$

then between the corresponding states $C^{(\alpha)} = \{u_i^{(\alpha)}, \psi_{ij}^{(\alpha)}, \varepsilon_{ij}^{(\alpha)}, \gamma_{ij}^{(\alpha)}, \chi_{ijk}^{(\alpha)}, \tau_{ij}^{(\alpha)}, \sigma_{ij}^{(\alpha)}, \mu_{ij}^{(\alpha)}\}$ there is the following reciprocity relation

$$(2.1) \quad \begin{aligned} & \int_0^t \int_{\Omega} (\rho f_i^{(1)}(x, t-\tau) u_i^{(2)}(x, \tau) + \rho l_{ij}^{(1)}(x, t-\tau) \psi_{ij}^{(2)}(x, \tau)) d\tau d\Omega + \\ & + \int_0^t \int_{\Sigma} (t_i^{(1)}(x, t-\tau) u_i^{(2)}(x, \tau) + T_{ij}^{(1)}(x, t-\tau) \psi_{ij}^{(2)}(x, \tau)) d\tau d\sigma = \\ & = \int_0^t \int_{\Omega} (\rho f_i^{(2)}(x, t-\tau) u_i^{(1)}(x, \tau) + \rho l_{ij}^{(2)}(x, t-\tau) \psi_{ij}^{(1)}(x, \tau)) d\tau d\Omega + \\ & + \int_0^t \int_{\Sigma} (t_i^{(2)}(x, t-\tau) u_i^{(1)}(x, \tau) + T_{ij}^{(2)}(x, t-\tau) \psi_{ij}^{(1)}(x, \tau)) d\tau d\sigma. \end{aligned}$$

Proof. By applying Laplace transformation with respect to t [3] Eqs (1.1)–(1.3) and (1.6) in homogeneous initial conditions, we get

$$(2.2) \quad (\tau_{ji} + \sigma_{ji})_{,j} + \rho f_i = \rho p^2 u_i, \quad \mu_{kij,k} + \bar{\sigma}_{ij} + \rho \bar{l}_{ij} = I_{is} p^2 \bar{\psi}_{sj},$$

$$(2.3) \quad 2\bar{\varepsilon}_{ij} = \bar{u}_{i,j} + \bar{u}_{j,i}, \quad \bar{\gamma}_{ij} = \bar{u}_{j,i} - \bar{\psi}_{ij}, \quad \bar{\chi}_{ijk} = \bar{\psi}_{jk,i},$$

$$(2.4) \quad \bar{\tau}_{ij} = C_{ijkl} \bar{\varepsilon}_{kl} + G_{klij} \bar{\gamma}_{kl} + F_{pqrij} \bar{\chi}_{pqr}$$

$$\bar{\sigma}_{ij} = G_{ijkl} \bar{\varepsilon}_{kl} + B_{klij} \bar{\gamma}_{kl} + D_{ijkpqr} \bar{\chi}_{pqr},$$

$$\bar{\mu}_{ijk} = F_{kajrs} \bar{\varepsilon}_{sr} + D_{rsijk} \bar{\gamma}_{rs} + A_{ijkpqr} \bar{\chi}_{pqr},$$

$$(2.5) \quad \begin{aligned} \bar{u}_i &= \bar{\tilde{u}}_i \quad \text{on } \Sigma_1 \times [0, p_1), \quad \bar{t}_i = (\bar{\tau}_{ji} + \bar{\sigma}_{ji})n_j = \bar{\tilde{t}}_i \quad \text{on } \Sigma_2 \times [0, p_1), \\ \bar{\psi}_{ij} &= \bar{\tilde{\psi}}_{ij} \quad \text{on } \Sigma_3 \times [0, p_1), \quad \bar{T}_{ij} = \bar{\mu}_{kij}n_k = \bar{\tilde{T}}_{ij} \quad \text{on } \Sigma_4 \times [0, p_1). \end{aligned}$$

On the basis of relations (2.4), (1.4) we get

$$(2.6) \quad \bar{\tau}_{ij}^{(1)} \bar{\epsilon}_{ij}^{(2)} + \bar{\sigma}_{ij}^{(1)} \bar{\gamma}_{ij}^{(2)} + \bar{\mu}_{ijk}^{(1)} \bar{\kappa}_{ijk}^{(2)} = \bar{\tau}_{ij}^{(2)} \bar{\epsilon}_{ij}^{(1)} + \bar{\sigma}_{ij}^{(2)} \bar{\gamma}_{ij}^{(1)} + \bar{\mu}_{ijk}^{(2)} \bar{\kappa}_{ijk}^{(1)}.$$

If we introduce the notation

$$(2.7) \quad K_{\alpha\beta} = \int_{\Omega} (\bar{\tau}_{ij}^{(\alpha)} \bar{\epsilon}_{ij}^{(\beta)} + \bar{\sigma}_{ij}^{(\alpha)} \bar{\gamma}_{ij}^{(\beta)} + \bar{\mu}_{ijk}^{(\alpha)} \bar{\kappa}_{ijk}^{(\beta)}) d\Omega,$$

from (2.6) we have

$$(2.8) \quad K_{12} = K_{21}.$$

Using the relations (2.3), (2.2), (2.7) and the divergence theorem, we obtain

$$(2.9) \quad \begin{aligned} K_{\alpha\beta} &= \int_{\Sigma} (\bar{t}_i^{(\alpha)} \bar{u}_i^{(\beta)} + \bar{T}_{ij}^{(\alpha)} \bar{\psi}_{ij}^{(\beta)}) d\sigma + \int_{\Omega} [(\rho \bar{f}_i^{(\alpha)} - \rho p^2 \bar{u}_i^{(\alpha)}) \bar{u}_i^{(\beta)} + (\rho \bar{l}_{ij}^{(\alpha)} - \\ &\quad - I_{is} p^2 \bar{\psi}_{is}^{(\alpha)}) \bar{\psi}_{ij}^{(\beta)}] d\Omega. \end{aligned}$$

From (2.8), (2.9), (1.4) and T210 [3] we obtain the reciprocity relation (2.1). This theorem, without using Laplace transform were established in [2].

In the quasi-static linear theory of elasticity with microstructure we obtain the following reciprocity relation

$$(2.10) \quad \begin{aligned} &\int_{\Omega} (f_i^{(1)} u_i^{(2)} + l_{ij}^{(1)} \psi_{ij}^{(2)}) d\Omega + \int_{\Sigma} (t_i^{(1)} u_i^{(2)} + T_{ij}^{(1)} \psi_{ij}^{(2)}) d\sigma = \\ &= \int_{\Omega} (f_i^{(2)} u_i^{(1)} + l_{ij}^{(2)} \psi_{ij}^{(1)}) d\Omega + \int_{\Sigma} (t_i^{(2)} u_i^{(1)} + T_{ij}^{(2)} \psi_{ij}^{(1)}) d\sigma. \end{aligned}$$

3. Variational Theorem. If we define as an admissible state $S = \{u_i, \psi_{ij}, \epsilon_{ij}, \gamma_{ij}, \kappa_{ijk}, \tau_{ij}, \sigma_{ij}, \mu_{ijk}\}$ the ordered assembly of functions $u_i, \psi_{ij}, \epsilon_{ij}, \gamma_{ij}, \kappa_{ijk}, \tau_{ij}, \sigma_{ij}, \mu_{ijk}$ defined on $\bar{\Omega} \times [0, t)$ with the properties

$$u_i \in C^{1,2}; \quad \psi_{ij} \in C^{1,2}; \quad \epsilon_{ij}, \gamma_{ij}, \kappa_{ijk} \in C^{0,0}, \quad \tau_{ij}, \sigma_{ij}, \mu_{ijk} \in C^{1,0}.$$

If we define the addition of the admissible states and the multiplication of an admissible state by a scalar as

$$\begin{aligned} S + S' &= \{u_i + u'_i, \psi_{ij} + \psi'_{ij}, \epsilon_{ij} + \epsilon'_{ij}, \gamma_{ij} + \gamma'_{ij}, \kappa_{ijk} + \kappa'_{ijk}, \tau_{ij} + \tau'_{ij}, \\ &\quad \sigma_{ij} + \sigma'_{ij}, \mu_{ijk} + \mu'_{ijk}\}, \end{aligned}$$

$$\alpha S = \{\alpha u_i, \alpha \psi_{ij}, \alpha \epsilon_{ij}, \alpha \gamma_{ij}, \alpha \kappa_{ijk}, \alpha \tau_{ij}, \alpha \sigma_{ij}, \alpha \mu_{ijk}\},$$

we find out that the set of all admissible states is a linear space. We shall define a cinematic admissible state as an admissible state satisfying Eqs. (1.2), (1.3) as well as the conditions imposed on the boundary part Σ_1 and Σ_3 . We call solution of the problem an admissible state satisfying Eqs. (1.1)–(1.3), initial conditions (1.5) and boundary conditions (1.6).

Consider $A = \{u_i, \psi_{ij}, \epsilon_{ij}, \gamma_{ij}, \kappa_{ijk}, \tau_{ij}, \sigma_{ij}, \mu_{ijk}\}$ a cinematic admissible state and the functional

$$\begin{aligned}
 F(A) = & \frac{1}{2} \int_{\Omega} (C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + B_{ijkl} \gamma_{ij} \gamma_{kl} + A_{ijpqr} \kappa_{ijk} \kappa_{pqr} + \\
 (3.1) \quad & + 2G_{ijkl} \gamma_{ij} \varepsilon_{kl} + 2F_{ijkrs} \kappa_{ijk} \varepsilon_{rs} + 2D_{ijpqr} \gamma_{ij} \kappa_{pqr}) d\Omega - \\
 & - \int_{\Omega} (\rho f_i u_i + \rho l_{ij} \psi_{ij}) d\Omega - \int_{\Sigma_2} \tilde{t}_i u_i d\sigma - \int_{\Sigma_4} \tilde{T}_{ij} \psi_{ij} d\sigma
 \end{aligned}$$

defined on set K of the cinematic admissible state.

The elastic energy density for unit volume $\mathcal{A}$ has the form

$$\begin{aligned}
 \mathcal{A}(\varepsilon_{ij}, \gamma_{ij}, \kappa_{ijk}) = & \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} B_{ijkl} \gamma_{ij} \gamma_{kl} + \frac{1}{2} A_{ijklmn} \kappa_{ijk} \kappa_{lmn} + \\
 & + D_{ijklm} \gamma_{ij} \kappa_{klm} + F_{ijklm} \kappa_{ijk} \varepsilon_{lm} + G_{ijkl} \gamma_{ij} \varepsilon_{kl}.
 \end{aligned}$$

Theorem 3.1 [1]. *If we suppose $\mathcal{A}(\varepsilon_{ij}, \gamma_{ij}, \kappa_{ijk})$ is a quadratic positive definite form and S is a solution to the mixed problem in static case, then*

$$(3.2) \quad F(S) \leq F(A), \quad \forall A \in K.$$

Proof. Consider $S, A \in K$ and $A' = A - S$. We have

$$(3.3) \quad 2\varepsilon'_{ij} = u'_{i,j} + u'_{j,i}, \quad \gamma'_{ij} = u'_{j,i} - \psi'_{ij}, \quad \kappa'_{ijk} = \psi'_{jki},$$

$$(3.4) \quad \tau'_{ij} = C_{ijkl} \varepsilon'_{kl} + G_{klij} \gamma'_{kl} + F_{pqrij} \kappa'_{pqr},$$

$$(3.5) \quad \sigma'_{ij} = G_{ijkl} \varepsilon'_{kl} + B_{klij} \gamma'_{kl} + D_{ijpqr} \kappa'_{pqr},$$

$$\mu'_{ijk} = F_{ijkrs} \varepsilon'_{rs} + D_{rsijk} \gamma'_{rs} + A_{ijkpqr} \kappa'_{pqr};$$

$$(3.5) \quad u'_i = 0 \text{ on } \Sigma_1, \quad \psi'_{ii} = 0 \text{ on } \Sigma_3.$$

Taking into account (1.4) and (1.3) the functional (3.1) can be put in the form

$$\begin{aligned}
 F(A) = & F(S) + \int_{\Omega} U' d\Omega + \int_{\Omega} (\tau'_{ij} \varepsilon'_{ij} + \sigma'_{ij} \gamma'_{ij} + \mu'_{ijk} \kappa'_{ijk}) d\Omega - \\
 (3.6) \quad & - \int_{\Omega} (\rho f_i u'_i + \rho l_{ij} \psi'_{ij}) d\Omega - \int_{\Sigma_2} \tilde{t}_i u'_i d\sigma - \int_{\Sigma_4} \tilde{T}_{ij} \psi'_{ij} d\sigma,
 \end{aligned}$$

where

$$\begin{aligned}
 (3.7) \quad 2U' = & C_{ijkl} \varepsilon'_{ij} \varepsilon'_{kl} + B_{ijkl} \gamma'_{ij} \gamma'_{kl} + A_{ijklmn} \kappa'_{ijk} \kappa'_{lmn} + \\
 & + 2D_{ijklm} \gamma'_{ij} \kappa'_{klm} + 2F_{ijklm} \kappa'_{ijk} \varepsilon'_{lm} + 2G_{ijkl} \gamma'_{ij} \varepsilon'_{kl}.
 \end{aligned}$$

Replacing (3.3) in (3.6) and applying the divergence theorem, taking into account (3.5) and grouping correspondingly, we get

$$\begin{aligned}
 (3.8) \quad F(A) = & F(S) + \int_{\Omega} U' d\Omega - \int_{\Omega} [(\tau_{ji} + \sigma_{ji})_{,j} + \rho f_i] u'_i d\Omega - \int_{\Omega} (\mu_{kij,k} + \sigma_{ij} + \\
 & + \rho l_{ij}) \psi'_{ij} d\Omega + \int_{\Sigma_2} [(\tau_{ji} + \sigma_{ji}) n_j - \tilde{t}_i] u'_i d\sigma + \int_{\Sigma_4} (\mu_{kij} n_k - \tilde{T}_{ij}) \psi'_{ij} d\sigma.
 \end{aligned}$$

S being a solution of the mixed problem in the case of elastic equilibrium from (3.8) it results

$$(3.9) \quad F(A) = F(S) + \int_{\Omega} U' d\Omega.$$

The elastic energy density $\mathcal{A}$ is a quadratic positive definite form i.e. there exists a number $c > 0$ such that for all $x \in \Omega$ there holds

$$(3.10) \quad U \geq c \sum_{i,j,k=1}^3 (\epsilon_{ij}^2 + \gamma_{ij}^2 + \chi_{ijk}^2).$$

Using (3.10) in (3.9), we have

$$(3.11) \quad F(S) \leq F(A), \quad \forall A \in K,$$

$$(3.12) \quad F(S) = F(A) \Leftrightarrow \epsilon'_{ij} = 0, \quad \gamma'_{ij} = 0, \quad \chi'_{ijk} = 0.$$

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REFERENCES

1. Ieșan D., *Mecanica generalizată a solidelor*. Univ. „Al. I. Cuza” Iași, 1980.
2. Ieșan C. M., *Sur quelques théorèmes généraux dans la théorie linéaire des milieux élastiques avec microstructure*. C. Rend. Acad. Sc. Paris **270**, 960–962 (1970).
3. Corduneanu A., *Ecuatii diferențiale cu aplicații în electrotehnică*. Ed. „Facla”, Timișoara, 1981.
4. Soos E., *Reciprocity Theorems for Elastic and Thermo-elastic Materials with Microstructure*. Bull. Acad. Polon. Sci., Ser. Sci. Techn., **16**, 895–900 (1968).

O TEOREMĂ DE RECIPROCITATE ȘI O TEOREMĂ VARIATIONALĂ PENTRU MEDII ELASTICE CU MICROSTRUCTURĂ

(Rezumat)

Se prezintă o teoremă de reciprocitate folosind transformata Laplace și o teoremă variatională pentru medii elastice cu microstructură.

The above information is a summary of the data collected from the survey conducted in the month of June 1961.

1.00	$\sum_{i=1}^n x_i$
1.01	$\sum_{i=1}^n x_i^2$
1.02	$\sum_{i=1}^n x_i^3$
1.03	$\sum_{i=1}^n x_i^4$
1.04	$\sum_{i=1}^n x_i^5$
1.05	$\sum_{i=1}^n x_i^6$
1.06	$\sum_{i=1}^n x_i^7$
1.07	$\sum_{i=1}^n x_i^8$
1.08	$\sum_{i=1}^n x_i^9$
1.09	$\sum_{i=1}^n x_i^{10}$
1.10	$\sum_{i=1}^n x_i^{11}$

The above information is a summary of the data collected from the survey conducted in the month of June 1961. The data is presented in the form of a table with 11 rows and 2 columns. The first column contains the row number and the second column contains the sum of the 11th power of the data values.

APPENDIX A

The above information is a summary of the data collected from the survey conducted in the month of June 1961.

MATHEMATICAL MODELLING OF THE RELATION BETWEEN THE Ca ION CONTENT OF FERRITE IN THE $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ SYSTEM AND SOME ELECTRICAL PROPERTIES

BY

ANASTASIA PĂDURARU, TRAIAN VIDRAȘCU, M. BARBU and EMIL LUCA

1. Theoretical Considerations. For obtaining experimental data on the variation of resistivity (conductivity) and on the energy of activation of conductivity, for the compounds of $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ system, as a function of the concentration of Ca ions x , ferrites of stoichiometric composition, synthesized at 1300°C and cooled slowly (at one time, with the oven, at a rate of 1.5°C/min) have been used.

For preparing the ferrites a procedure similar to that used for obtaining most of the ceramics has been applied [1] but with a more severe control of the technology used to obtain the desired mechanical, electrical and magnetical properties.

The resistivity of the ferrite samples was measured by means of a Wheatstone-Thomson bridge with an accuracy of $\pm 0.02\%$, at a constant temperature (27°C), obtained with a thermostated oven which assured an accuracy of $\pm 0.05^\circ\text{C}$.

In Fig. 1 the variation of resistivity as a function of the concentration x of Ca ions, in the stoichiometric compound $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ synthesized at 1300°C for six hours and cooled slowly at a rate of 1.5°C/min, is plotted in logarithmic coordinates $\lg \rho = f(x)$.

As can be seen in Fig. 1, for small concentrations, that is with in $0 < x < 0.05$ range a small increase of the resistivity is noticed. For x values higher than 0.05 ($x > 0.05$) a marked decrease of the resistivity with increasing content of Ca ions is found, up to a concentration $x \approx 0.15$. A higher values of the Ca ion content, that is for $x > 0.15$, the resistivity increases

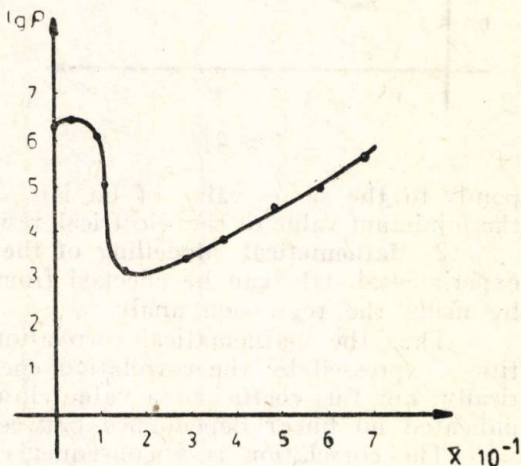


Fig. 1

with increasing x . For $x \approx 0.15$ the ferrite exhibits a minimum of the $\lg \rho$ quantity. The $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ system, synthesised at 1300°C and cooled slowly exhibit a higher resistivity (less than an order of magnitude) than the NiFe_2O_4 ferrite, prepared under the same conditions, excepting the concentration of Ca ions which was smaller ($x < 0.1$).

Along with the investigations on some electrical properties of the ferrites belonging to the $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ system the variation of the ferrite resistivity with temperature has been followed. The measurements were taken within $30^\circ < t < 200^\circ\text{C}$ temperature range. By plotting $\lg \rho = f(1/T)$ the activation energy of the conductivity q has been determined using the relation

$$(1) \quad q = 0.198 \cdot 10^{-3} \frac{d(\lg \rho)}{d(1/T)}.$$

The values obtained for the activation energy of the conductivity q , for various spinel ferrites, comprised between 0.07 and 0.31 eV are in good agreement with those reported in the literature for spinel ferrites.

Since the value of energy of activation depends on the concentration x of Ca ions in the $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ ferrites, as well as on the cooling manner,

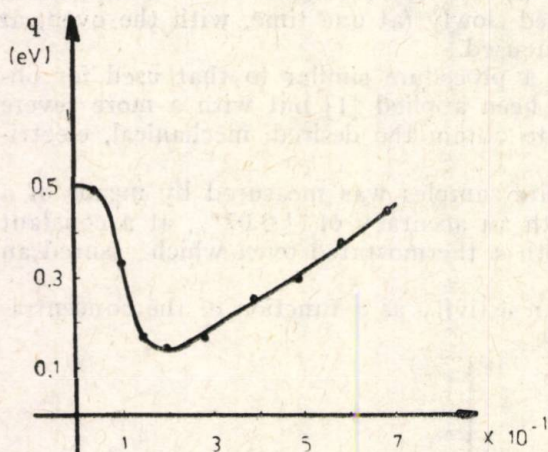


Fig. 2

the curves of variation of activation energy of conductivity against the Ca ion cooled concentration x , for slowly cooled ferrites have been plotted in Fig. 2. By analysing the obtained plot it can be seen that with the slowly cooled ferrite the energy of activation of conductivity decreases initially with the concentration x of Ca ions, till a minimum value, for $x \approx 15$, and increases then for $x > 0.15$ values.

The minimum value of the energy of activation of electrical conductivity corresponds

to the same value of Ca ion concentration ($x \approx 0.15$) at which the minimum value of the electrical resistivity of the same ferrite occurs.

2. Mathematical Modelling of the Phenomena. All of the obtained experimental data can be checked from the mathematical point of view by using the regression analysis.

Thus the mathematical correlation between the investigated quantities, expressed by the correlation coefficient r , could be settled statistically. For this coefficient a value close to zero has been obtained which indicated no linear dependence between the two investigated variables.

The correlation is a consequence of the stochastic binding which consists in the fact that a variable reacts to the variation of the other, by

modifying the law of distribution [2]. This fact deserves mention since it is often considered that when the correlation ratio is zero there is no dependence between the two variables. In fact, this is not exact, since the variation of the x variable can influence the variation of the other variables without modifying the group center.

Hence, based on the mathematical statistics the correlation coefficients r have been calculated for the $\log \rho = f(x)$ dependence, for which the obtained r value was $r=0.02$ and the $q=f(x)$ dependence, for which $r=0.061$.

To investigate the statistical-mathematical dependence of the electrical resistivity as a function of the concentration of Ca ions in the ferrites of the $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ system the concentration of Ca ions was denoted by x and the quantity $\log \rho$ by y .

The experimental obtained values are summarized in Table 1. By means of least square method the regression $\log \rho = f(x)$ equation modelling the investigated phenomenon has been obtained as

$$(2) \quad \log \rho = 20.782 x^2 - 14.490 x + 6.203.$$

As can be seen the equation represents a second order polynomial function whose extreme point is given by $x=0.349$ and $y=3.68$.

It can be noticed that for $0.15 < x < 0.3$ y takes values comprised between 3.7 and 4.5 which represents only 17% of the maximum value of 4.5. Hence it may be stated that within this range y has values very close to the minimum value of the function. For checking up the obtained equation the calculated and found values are listed in Table 1.

To estimate the truthfulness of mathematical modelling the deviation of the theoretical value y_{theor} , with respect to the experimental one y_{exp} , was calculated according to the equation

$$(3) \quad \Delta = \frac{y_{\text{exp}} - y_{\text{theor}}}{y_{\text{exp}}} 100$$

and the obtained values were tabulated in Table 1.

By inspecting the data in Table 1 it can be noticed that only two out of the 15 runs exhibit deviations larger than 10% (15.3 and 10.8%, respectively). This allows the statement, that, according to Student, the phenomenon is mathematically modelled by an approximation of 90%. In Fig. 3 the variation of calculated and experimental y values as a function of x is plotted.

The mathematical-statistical dependence of the quantity q , the activation energy of electrical conductivity on the concentration of Ca ions in the ferrites of the $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ system was approached in a similar manner. The experimental data used are listed in Table 2.

The obtained regression equation has the form of a second order polynomial function

$$(4) \quad q = 2.419 x^2 - 1.611 y + 0.455$$

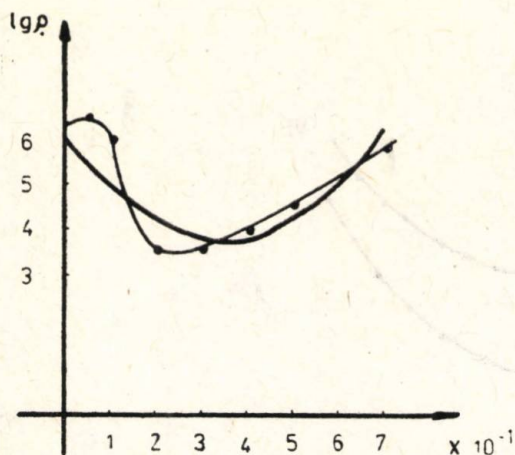
with the extreme point given by $x=0.333$ and $q=0.19$ eV.

Table 1

x	0.0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70
y_{exp}	6.3	6.4	5.9	4.1	3.7	3.5	3.6	3.8	4.0	4.2	4.6	4.9	5.3	5.2	6.20
y_{theor}	6.2	5.8	5.0	4.5	4.1	3.8	3.7	3.68	3.73	3.9	4.2	4.52	4.99	5.56	6.20
Δ (%)	1.6	9.4	15.3	9.8	10.8	8.6	2.8	3.2	6.8	7.1	8.7	7.8	5.8	6.9	0

Table 2

x	0.0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70
y_{exp}	0.5	6.4	0.32	0.21	0.20	0.18	0.18	0.20	0.24	0.26	0.29	0.32	0.38	0.41	0.48
y_{theor}	0.48	0.39	0.32	0.24	0.22	0.20	0.19	0.19	0.21	0.32	0.27	0.30	0.36	0.43	0.50
Δ %	4	7	0	14	10	11	5.6	5.0	12.5	23	6.9	6.3	5.3	4.9	4.2



$$\gamma = 20.782x^2 - 14.490x + 6.203$$

Fig. 3

For $0.15 < x < 0.3$, y takes values comprised between 0.20 and 0.26 which amounts to 23% of the maximum value, $y = 0.26$. To check up the obtained equation the calculated y_{theor} values have been tabulated and compared to the found values. The truthfulness of the obtained values is estimated by calculating the deviation Δ , according to (3).

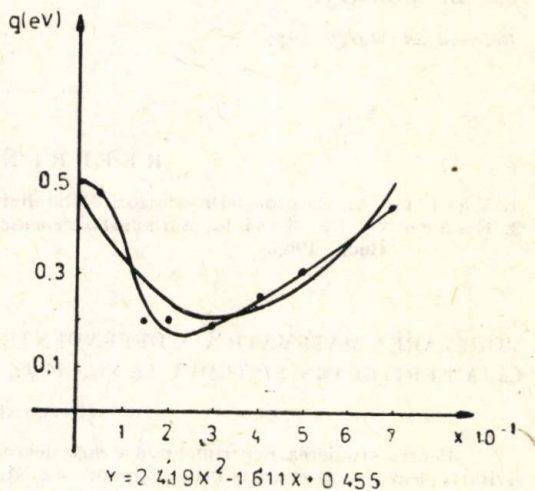
The inspection of data in Table 2 indicates that only three of 15 experiments exceed the 10 per cent deviation (14%, 11% and 12.5% deviation, respectively). It can be stated that the

phenomenon is modelled by an approximation of 10%.

In Fig. 4 the theoretical and experimental values of the quantity $y = q$ are listed. Taking into account the inherent deviations caused by the preparative procedures of ferrites in the technological process, the acceptable errors of the measuring and control apparatus, it may be affirmed that the process modelling at an approximation of 90% is satisfactorily good for industrial practice.

3. Conclusions. 1. To obtain a certain ferrite of the $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ system, with a certain resistivity or energy of activation of electrical conductivity, usually time and material consuming trials are required. By using the nomograph in Fig. 5 and the regression equation derived, for an imposed resistivity and energy of activation, the required concentration of Ca ions can be easily determined.

2. This method is faster and suppress the experimental trials which results in economies in raw materials and manual labour.



$$\gamma = 2.419x^2 - 1.611x + 0.455$$

Fig. 4

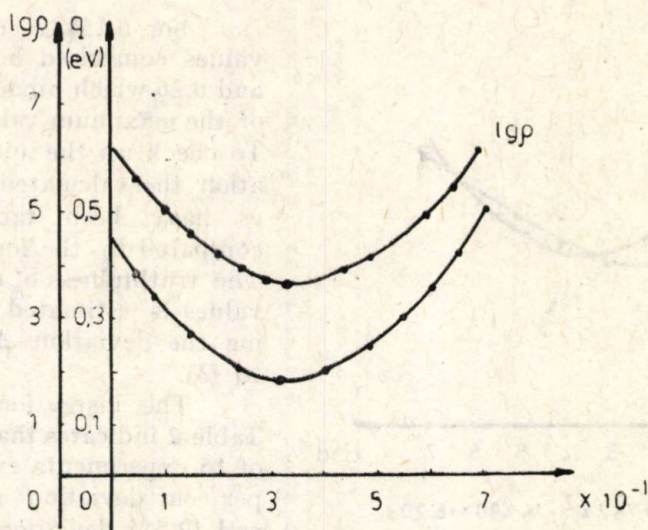


Fig. 5

3. Our research group intends to extend the study to all of the electrical and magnetical properties specific to the ferrites of this system so that to be able to elaborate a more complete nomograph with practical use in industry.

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REFERENCES

1. Vasiliu A., Doctoral Dissertation, Polytechnic Institute Jassy, 1978.
2. Rancu N., Tövissi L., *Statistică matematică cu aplicații în producție*. Ed. Acad. R.P.R., Buc., 1963.

MODELAREA MATEMATICĂ A DEPENDENȚEI DINTRE CONȚINUTUL ÎN IONI DE Ca A FERITEI DIN SISTEMUL $\text{Ca Ni}_{1-x} \text{Fe}_2\text{O}_4$ ȘI UNELE PROPRIETĂȚI ELECTRICE

(Rezumat)

Pentru studierea experimentală a variației rezistivității și a energiei de activare a conductivității electrice, la compuşii din sistemul $\text{Ca}_x\text{Ni}_{1-x}\text{Fe}_2\text{O}_4$ în funcție de concentrația în ioni de Ca x , s-au utilizat ferite de compoziție stoechiometrică, sinterizate la 1300°C și răcite lent (odată cu cuptorul cu viteza de 1,5°C/min).

Toate datele experimentale obținute au fost verificate din punct de vedere matematic, utilizând analiza de regresie. Astfel s-au stabilit ecuațiile de regresie corespunzătoare și s-a trasat o nomogramă, cu care se pot determina direct mărimea concentrației în ioni de Ca necesare, pentru o anumită rezistivitate și energie de activare impuse.

Metoda este mai rapidă și suprimă total experimentele, ceea ce determină economii de materii prime și manoperă.

ON SOME CHARACTERISTICS OF THE SCHOTTKY SOLAR CELLS

BY

SOFIA MELINTE, DAN MELINTE, ALEXANDRINA JEFLEA and MARCEL MOISE

1. Introduction. In the present development stage, when large application of photovoltaic systems becomes a probable possibility, the lowering of the price of cost represents a major problem. The price of a cell depends, among other things, on the abundance of the used materials, on their melting points which determine the energy consumption for the structure production process, and on the production technology.

The celles based on hetero- and $p-n$ junctions, especially those based on monocrystals, have proved their efficiency in solar energy conversion, but their price is still high.

For the terrestrial applications, the only possibility to lower the cost consists in the simplification of the production process. That is why the idea of a metal-semiconductor junction seems very attractive at the first sight. At the same time, the metal deposition methods are compatible with large areas, thus preventing the semiconductor degradation, which also suggests the utilization of the polycrystalline substrate. The Schottky photovoltaic structures can be made of a n -type semiconductor with a high work function metal, like Au, Ag, Pt, Cr etc., or a p -type semiconductor and a low work function metal. More frequent are the structures with n -type semiconductors, especially on monocrystals of Si, GaAs, CdS, and structures on polycrystalline layers are rare.

In order that such a cell should have a high conversion gain, it is necessary that the Schottky barrier height to be as high as possible, close to the energetic value of the semiconductor forbidden band (the „gap“).

In this work the temperature dependence is discussed for different parameters affecting the cell efficiency, especially for the Schottky barrier height.

2. Experimental Procedure. CdS-based Schottky structure was realized by a vacuum evaporation on a heated substrate. The Schottky barrier is of CdS/Au-type.

The ohmic contact between CdS and Cu metallic substrate was made by the interposition of a Zn layer. Both the thicknesses of CdS and Au layers were estimated interferometrically. It was established by X-ray diffraction that the deposited CdS layers are polycrystalline and of n -type. In order to determine the Schottky barrier height, measurements of the dark current were made at different temperatures, depending on the applied

voltage, that ranged between 0.1 and 0.5 V. From the $I-V$ dark characteristics, given by the relation

$$(1) \quad I_{\text{dark}} = I_0 \exp \frac{eV}{nkT} - 1 \simeq I_0 \exp \frac{eV}{nkT}$$

and presented in Figs. 1...3, the preexponential factor I_0 and the barrier height were determined with

$$(2) \quad \Phi_b = \frac{kT}{e} \ln \left(\frac{AT^2}{I_0} \right),$$

where: $A = 28.8 \text{ A/cm}^2\text{grad}^2$ is Richardson's constant, determined for $m^* = 0.2 m_e$ in the case of CdS; k is Boltzmann's constant; T is the absolute temperature; e is the electron charge.

The barrier height is given by the surface state at the metal-semiconductor interface or, in the case of the interposition of an oxide, between this semiconductor and the metal. The insulating (oxide) layer which is very thin (some Angströms) has the role of suppressing the current given by the majority carriers. In a structure of this type, an inversion layer is created [2], [5] that plays an important role in the amplification of the efficiency of the conversion of the light energy into the electric energy.

3. Results and Discussion. The phenomena that occur at the metal-semiconductor interface are generally known; yet, we must remind that, in the case of a metal with a work function higher than that of a n -type semiconductor, $\Phi_M > \Phi_s$, the energetic diagram from Fig. 4 is true, where the appearance of a double layer is obvious.

The departure of the electron from the semiconductor leaves the donor ions electrically unsatisfied, which produces a positive space charge close to the contact surface and the electrons from the metal are attracted toward this positive charge. Within this double layer an electric field is generated and the potential difference of the double layer is called the „Schottky barrier“. In the absence of the external field or the light radiation, the current through the barrier is zero. It follows that the two—drift and diffusion—currents are equal and opposed to each other.

Under the action of a light radiation with the energy $h\nu \geq E_g$, the electron-hole pairs generated within the Schottky barrier can be separated due to the local field, so that they will generate a voltage. The generated voltage depends on the potential barrier height, which in turn depends on the difference between the work functions of the two materials. Yet, actually this metal-semiconductor contact presents a great shortcircuiting, namely a low photovoltaic response. This consists in the fact that the thermionic emission of the dark current is considerably higher than in the $p-n$ junction.

The theory and the experiment [3], [4] show that an increase of the efficiency can be obtained by deliberately introducing an insulating layer between the metal and the semiconductor, which leads to the realization of a MIS cell [5]. In these cells, the dark current can be reduced by the effective increase of the barrier height, thus reducing the probability of the ma-

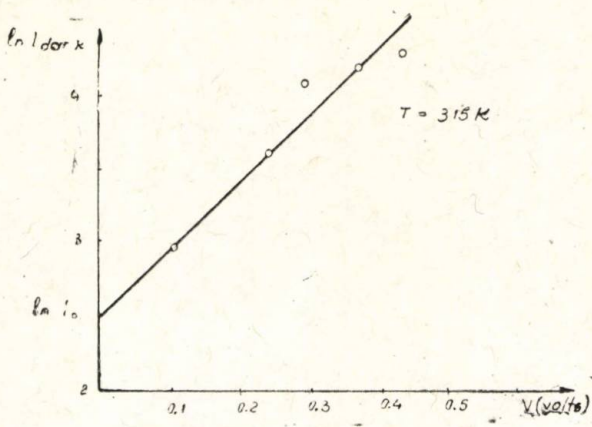


Fig. 1 — $\ln I_{\text{dark}}$ vs. applied V , for CdS/Au Schottky cell at $T = 315 \text{ K}$.

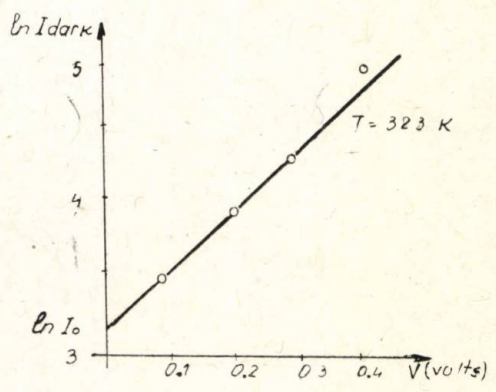


Fig. 2 — $\ln I_{\text{dark}}$ vs. applied V , for CdS/Au Schottky cell at $T = 323 \text{ K}$.

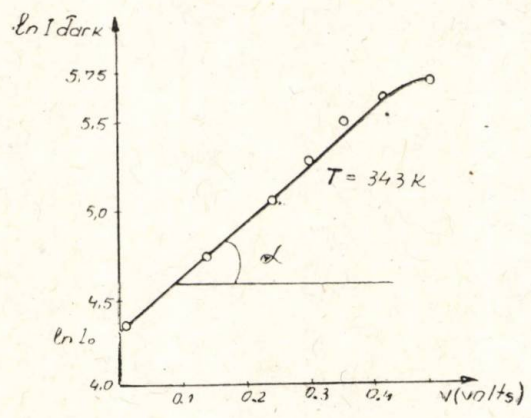
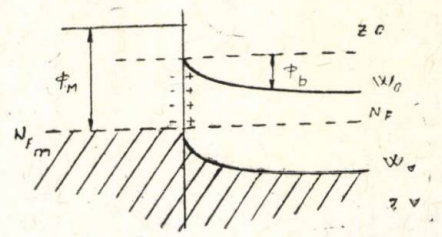


Fig. 3 — $\ln I_{\text{dark}}$ vs. applied V , for CdS/Au Schottky cell at $T = 343 \text{ K}$.

Fig. 4 — Energetic diagram at metal- n -type semiconductor interface; $\Phi_M > \Phi_S$.



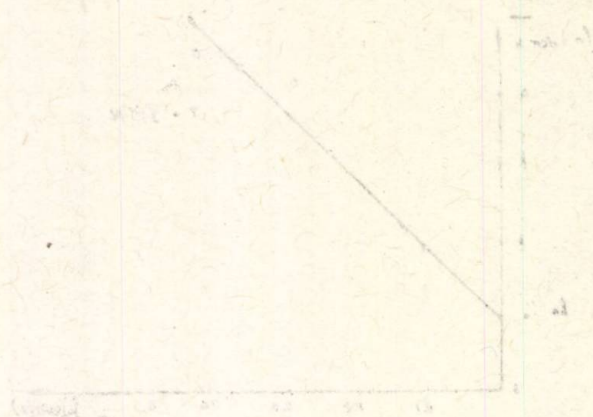


Fig. 1 - In I-V characteristic for CdS/Al Schottky cell at $T = 312$ K.

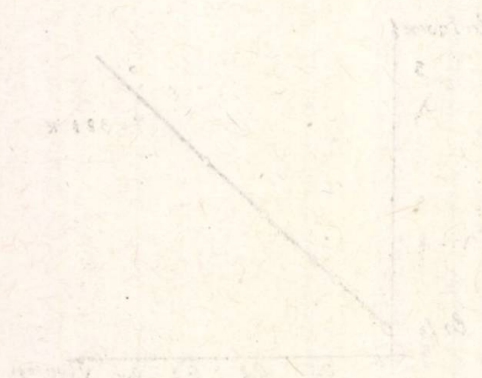


Fig. 2 - In I-V characteristic for CdS/Al Schottky cell at $T = 325$ K.

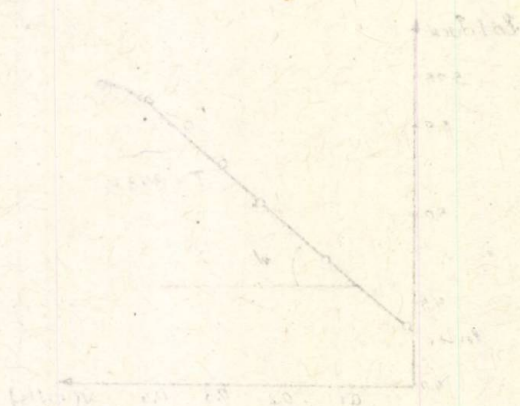


Fig. 3 - In I-V characteristic for CdS/Al Schottky cell at $T = 340$ K.



Fig. 4 - Energy level diagram at metal-n-type semiconductor interface; $\Phi_n > \Phi_m$.

majority carriers to penetrate the barrier, keeping a number of capture centers in the interface states. In this case, the total density of the dark current is more complex than in the case of a $p-n$ junction, given by the relation

$$(3) \quad j = j_t + j_{rg} + j_a + j_s,$$

where j_t is the current density due to the hole thermoionic emission within the metal; j_{rg} — the current density from the space charge layer; j_a — the injection-diffusion current density; j_s — the current density through the surface states due to the charge exchange between the metal and the semiconductor.

The explicit relation of the current density is

$$(4) \quad j = AT^2 \exp\left(-\frac{e\Phi_b}{kT}\right) \exp \frac{eV}{nkT},$$

where we can write

$$(5) \quad j_0 = AT^2 \exp\left(-\frac{e\Phi_b}{kT}\right).$$

By the graphical transposition of this relation (Fig. 5) under the form

$$\ln \frac{j_0}{AT^2} = f \frac{10^3}{T},$$

a straight line was obtained, from whose slope the barrier height $\Phi_b = 0.69$ eV was found.

By comparing this value with the values found on the basis of the relation (2), which range between $[0.77...0.68]$ eV, one can conclude that the CdS/Au Schottky barrier height is low in the both procedures.

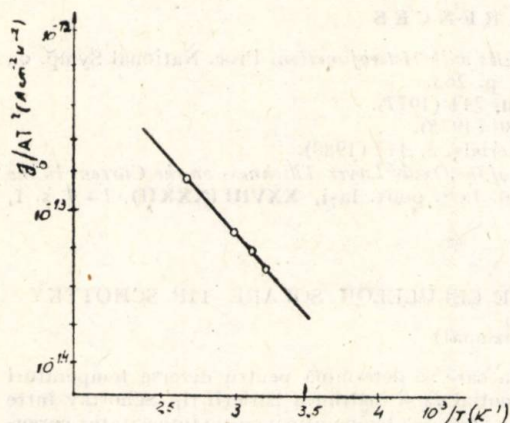


Fig. 5 — $\ln j_0/AT^2$ vs. $10^3/T$ for Schottky CdS/Au.

From both, the graphics given in Figs. 1...3 and the Table 1, it follows that the dark current density of a Schottky barrier increases with the temperature increase, which will determine a low short-circuit current at not very high temperatures, like 323 K or 343 K.

Table 1
Temperature dependence of I_0 and j_0

T (K)	$I_0 \cdot 10^{-6}$ A	$j_0 \cdot 10^{-6}$ A/cm ²
305	7.38	2.05
315	12.18	3.38
323	23.33	6.48
343	77.48	21.52

From the $I-V$ semilogarithmical characteristics given in Fig. 3, for $T=343$ K, one can notice that for a voltage exceeding 0.5 V this characteristic is no longer linear, which indicates that the series resistance is limited by the applied voltage.

4. Conclusions. 1. In the CdS/Au Schottky barriers the dark current has higher values and increases with increasing temperature.

2. The barrier height has lower values, much too low in comparison with the energetical value of the forbidden band of the CdS-type semiconductor, which is $E_g=2.4$ eV.

3. With the view of increasing the efficiency, the increase of the barrier is necessary by the interposition of an insulating layer between the metal and the semiconductor, this reducing the dark current and increasing the conversion efficiency by the increase of the Schottky barrier height.

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REFERENCES

1. Melinte S. et all., *CdS/Cu₂S Solar Cells with Heterojunction*. Proc. National Symp. on Solar Energy Utilization, 1980, p. 263.
2. Olsen L. C., *Solid State Electron.*, 20, 741 (1977).
3. Fonash S. J., *J. Appl. Phys.*, 46, 1286 (1975).
4. Roger J. A. et all., *Solar Energy Materials*, 2, 447 (1980).
5. Melinte S., Jeflea Al., *Influence of the Oxide Layer Thickness on the Current in the Schottky Barrier Solar Cells*. Bul. Inst. polit. Iași, XXVIII (XXXII), 1-4, s. I, 226 (1982).

ASUPRA UNOR CARACTERISTICI ALE CELULELOR SOLARE TIP SCHOTTKY

(Rezumat)

Se prezintă tipul de celulă CdS/Au la care se determină pentru diverse temperaturi din domeniul 300...343 K factorul preexponențial I_0 și înălțimea barierei tip Schottky între semiconductor și metal. Se observă că odată cu creșterea temperaturii crește intensitatea curentului prin celulă. Înălțimea barierei este scăzută ($\Phi_b = 0,69$ eV) ceea ce arată că pentru creșterea eficienței celulei este necesară mărirea barierei prin introducerea deliberată a unui strat izolator între semiconductor și metal, cu efect de tunelare a curentului.

UNE NOUVELLE RELATION POUR $\cos \theta$

PAR

AL. DINCĂ

1. Introduction. Pour calculer le densité de la puissance radiante de l'énergie solaire directe, sur une surface inclinée, on utilise la relation $E_i = E_a \cos \theta$, dans laquelle E_i est la densité de la puissance radiante sur une surface inclinée sur le plan horizontal; E_a est la densité de la puissance radiante incidente; θ est l'angle d'incidence de la radiation solaire; $\cos \theta$ est le facteur géométrique de la radiation solaire, qui fait l'objet de cette étude. Dans la fig. 1 on représente ces éléments [1].

L'Institut Météorologique de Bucarest possède des données pour E_a , en $\text{cal/cm}^2 \text{ min.}$ pour diverses régions de la Roumanie.

2. Le relation classique pour $\cos \theta$ est [1]

$$(1) \quad \cos \theta = \sin \delta (\sin \varphi \cos i - \cos \varphi \sin i \cos \gamma) + \\ + \cos \delta \cos H (\cos \varphi \cos i + \sin \varphi \sin i \cos \gamma) + \cos \delta \sin i \sin \gamma \sin H.$$

On applique les formules de Gauss à deux triangles sphériques de position et on obtient les relation suivantes, qui servent pour le déduction de la relation générale montrée :

$$\sin \delta = \sin \varphi \sin h - \cos \varphi \cos h \cos A; \quad \sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos H; \\ \sin A = \cos \delta \frac{\sin H}{\cos h}; \quad \cos \theta = \sin h \cos i + \cos h \sin i \cos (A - \gamma) \quad [2].$$

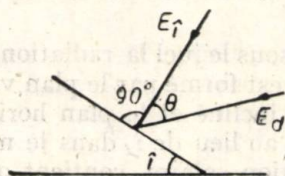


Fig. 1

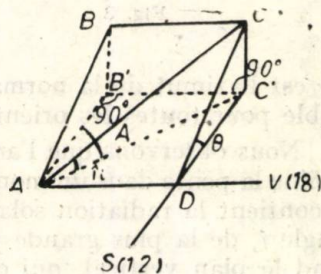


Fig. 2

3. Détermination de l'angle de pente i_i pour la pente orientée vers sud. Dans la fig. 2 est représentée une portion prismatique d'un terrain en pente, orientée vers sud, c'est à dire vers midi, noté par l'heure locale 12.À

midi la pente est illuminée de manière que la radiation solaire est parallèle avec le plan vertical ABB' , c'est à dire sous un angle i , de la plus grande pente. Dans un moment quelconque, après midi, la pente est illuminée

sous l'angle de pente $\widehat{CAC'} = i_i$, qui est de plus en plus petit, à mesure que le soleil avance vers coucher, de telle sorte que dans le moment quand le soleil illumine la pente parallèle au plan $BCC'B'$, $i_i = 0$. Les valeurs de i_i , avant midi, sont symétriques égales avec celles d'après midi. De cette figure résultent les relations $\text{tg } i_i = CC'/AC'$; $\text{tg } i = BB'/AB' = CC'/AB'$.

L'angle $\widehat{B'AC'}$ est l'angle plan, mesure de l'angle dièdre, qui est formé par le plan vertical, qui contient le plan méridien, avec le vertical qui contient

la radiation solaire, dans le moment H , donc $\widehat{B'AC'} = A$ (l'azimut du soleil, dans ce moment). De cette figure résulte encore $\cos A = BB'/AC'$. Mais $\text{tg } i_i$ peut être mise sous la forme $\text{tg } i_i = CC'/AC' = (CC'/AB') (AB'/AC')$. Substituant les valeurs de ces dernières rapports, nous obtenons

$$(2) \quad \text{tg } i_i = \text{tg } i \cos A.$$

4. Détermination de i_i pour une pente orientée vers SE. Dans la fig. 3 est représentée un terrain similaire à celui de la fig. 2, mais orienté vers

SE, correspondant à l'heure locale 9. Donc à cette heure le soleil illumine la pente parallèle au plan vertical ABB' , sous l'angle de la plus grande pente i . À l'heure locale $H=G$, le soleil illumine la pente de la direction E , parallèle au plan CAC' , sous l'angle de pente $\widehat{CAC'} = i_i$.

On est conduit facilement à la relation

$$(3) \quad \text{tg } i_i = \text{tg } i \cos (A - \gamma),$$

où γ est l'azimut de la normale au plan incliné $ABCD$. Cette relation est valable pour toutes les orientations δ .

Nous observons que l'angle de pente i_i , sous lequel la radiation solaire illumine la pente dans un moment quelconque, est formé par le plan vertical, qui contient la radiation solaire, avec le plan incliné et le plan horizontal. L'angle i , de la plus grande pente, intervient au lieu de i_i dans le moment quand le plan vertical, qui contient la radiation solaire, contient aussi la normale au plan incliné. Donc dans ce moment l'angle i_i a la valeur maximale i , d'où il diminue. L'angle $i_i = 0$ quand la radiation solaire est parallèle avec la horizontale contenue dans le plan incliné.

5. Les observations utilisées pour obtenir la nouvelle relation de $\cos \theta$. Ces observations sont :

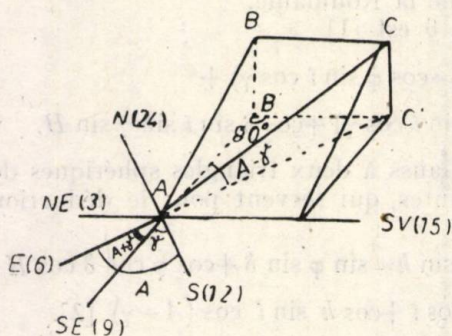


Fig. 3

a) Une surface plane inclinée peut être illuminée, pendant le jour, en deux modes : avec le soleil en face de la pente, c'est à dire par devant et avec le soleil en dos de la pente, c'est à dire par derrière. Par exemple une surface inclinée, orientée vers sud, est illuminée par la radiation solaire directe, à l'équinoxe, dans l'intervalle des heures locales 6 et 18, avec le soleil en face de la pente. Pendant l'été la même surface peut être illuminée, avec le soleil en arrière de la pente, dans les intervalles $R-6$ et $18-A$ (R et A représentant l'heure locale de lever et de coucher du soleil), avec la condition $i_i < h$, c'est à dire l'angle de pente i_i doit être inférieur à l'angle de la hauteur du soleil au dessus de l'horizon, dans ce moment.

b) Chaque manière d'illumination de la pente peut avoir une durée maximale de 12 heures.

c) Notant avec H_n l'heure locale, quand les rayons du soleil sont parallèles au plan vertical qui contient la normale au plan incliné, l'heure du commencement d'illumination par devant est $H_n - 6$ et l'heure de l'achèvement d'illumination par devant est $H_n + 6$, avec les conditions $H_n - 6 \geq R$ et $H_n + 6 \leq A$. Si la première condition n'est pas accomplie, on prend, au lieu de $H_n - 6$, l'heure de lever du soleil R . Si la deuxième condition n'est pas accomplie, on prend au lieu de $H_n + 6$, l'heure de coucher du soleil A .

d) La radiation solaire directe illumine une pente orientée vers quelque direction, sous un angle de pente i_i , décrit plus haut.

e) Au lieu de $\cos \theta$ on peut utiliser le sinus du complément de θ , c'est à dire $\sin(90^\circ - \theta)$.

6. La nouvelle relation pour $\cos \theta$. Dans les figs. 4a, 4b, 4c on montre le mode d'illumination de la radiation solaire directe sur une surface hori-

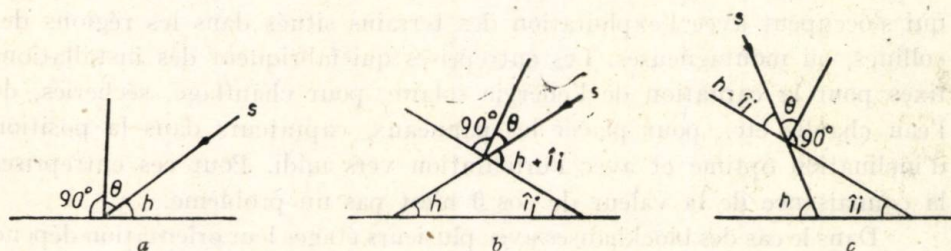


Fig. 4

zontale et sur une surface inclinée, avec le soleil en face et avec le soleil en arrière. Prenant au lieu de $\cos \theta$ son égal, qui est le sinus de son complément, on obtient dans le cas de la fig 4a $\cos \theta = \sin h$; dans le cas de la fig. 4b on observe que le complément de l'angle θ est extérieur au triangle formé, donc $\cos \theta = \sin(h + i_i)$. Dans le cas 4c le complément de l'angle θ a la valeur $h - i_i$, donc $\cos \theta = \sin(h - i_i)$.

La relation générale, qui contient les trois égalités, est

$$(4) \quad \cos \theta = \sin(h \pm i_i).$$

Dans le cas particulier de midi, quand nous avons deux surfaces inclinées, une surface orientée vers S et l'autre vers N, (4) devient $\cos \theta =$

$=\sin(h_m \pm i)$, dans laquelle h_m est la valeur de h à midi, donnée par l'expression $h_m = 90^\circ - \varphi \pm \delta$ (déterminée géométriquement) et i est l'angle de la plus grande pente. La valeur de h s'obtient par observation ou par calcul utilisant la relation classique (1). La valeur de i_i peut être déduite par (3). Utilisant la relation classique et la nouvelle relation, on obtient les mêmes résultats. La seule différence est que par la nouvelle relation, pour les moments de passage d'une sorte d'illumination à l'autre, on obtient deux valeurs : l'une pour l'achèvement d'une illumination et l'autre pour le commencement de l'autre illumination. La signification des symboles utilisés : φ — latitude géographique, H — angle horaire du soleil, δ — déclinaison du soleil, h — hauteur du soleil au dessus de l'horizon.

7. Les entreprises intéressées de connaître $\cos \theta$. Si dans $E_i = E_a \cos \theta$ on substitue la valeur horaire de E_a et la valeur horaire de $\cos \theta$, et on fait la somme des produits horaires pour l'intervalle d'illumination avec le soleil en face (intervalle défini au § 5a), à lequel on ajoute la somme des produits pour l'intervalle d'illumination avec le soleil en arrière (intervalle qui s'obtient de la différence entre l'intervalle journalier d'illumination, entre le lever et le coucher du soleil, et l'intervalle d'illumination avec le soleil en face de la pente) on obtient la valeur de l'énergie solaire arrivée au niveau de l'unité de surface captatrice, pendant le jour respectif. Comparant les résultats pour les diverses surfaces de captation, avec des orientations diverses, d'une même région, c'est à dire avec la même latitude et la même altitude, on peut étudier la variation du potentiel énergétique de l'unité de surface, avec la même inclination, pour les diverses orientations. Les calculateurs électroniques font le travail facile.

Une telle méthode de comparaison pourrait intéresser les entreprises qui s'occupent avec l'exploitation des terrains situés dans les régions des collines, ou montagneuses. Les entreprises qui fabriquent des installations fixes pour la captation de l'énergie solaire, pour chauffage, sécheries, de l'eau chaude etc., pour placer les panneaux captateurs dans la position d'inclination optimale et avec l'orientation vers midi. Pour ces entreprises la connaissance de la valeur de $\cos \theta$ n'est pas un problème.

Dans le cas des blockhauses avec plusieurs étages leur orientation dépend de la position des rues respectives, mais leur orientation peut être aussi vers midi, si le projet demande cette condition. Dans les rues avec des blockhauses paraît l'effet d'ombrage, qui empêche la pénétration de la radiation solaire directe. Pour éviter l'ombrage il faut que la distance en direction des rayons solaires, doit respecter la relation $d = I/\tan h$, où I est la hauteur du blockhaus de devant, et h — la hauteur du soleil au dessus de l'horizon, dans la moment respectif. La connaissance de $\cos \theta$ peut être nécessaire pour les constructions avec différentes orientations. Notons que les murs verticaux peuvent être illuminés seulement avec le soleil en face. L'angle d'incidence pendant l'hiver est le plus petit, donc les murs verticaux sont mieux échauffés pendant l'hiver. Les autres détails de cet étude seront présentés avec une autre occasion.

BIBLIOGRAPHIE

1. Dănescu Al., Bucurenciu S., Petrescu Șt., *Utilizarea energiei solare*. Ed. tehn., Buc., 1980.
2. Кондратиев К. И., Пиварова Л. С., Феодорова М. П., *Радиационный режим наклонных поверхностей*. Гидрометеориздат, Л., 1978.
3. Simionescu I., *Trigonometrie sferică*. Ed. tehn., Buc., 1965.
4. Pârvulescu C., *Complemente de astronomie*. Ed. did. și ped., Buc., 1967.
5. Stoica C., *Meteorologie generală*. Ed. tehn., Buc., 1978.
6. * * * *Anuarul astronomie 1982*. Ed. Acad. R.S.R., Buc., 1982.

O NOUĂ RELAȚIE PENTRU $\cos \theta$

(Rezumat)

Se stabilește o nouă relație pentru calculul factorului geometric al radiației solare, care facilitează aprecierea influenței înclinării suprafeței captatoare și a orientării sale, în scopul obținerii unei captări optime a energiei solare.

INFLUENCE OF SOLVENT UPON THE ABSORPTION ELECTRONIC SPECTRA OF SOME DERIVATIVES OF 4-AMIDO-SULPHONYL-PHENOXYACETIC ACID

BY

GEORGETA STRAT, C. GHIRVU, C. ONISCU, M. STRAT and V. NIȚICĂ

1. Introduction. The spectral methods used for studying some organic substances found in various medium and temperature conditions allow the obtaining of informations on both nature and magnitude of the intermolecular interaction forces manifested in various liquid solutions as well as on some microscopical parameters characterizing the molecule of investigated compound dissolved in various solvents.

In the present paper two derivatives of the 4-amidosulphonyl-phenoxyacetic acid are investigated by means of electronic absorption spectra. The compounds under study are: the 3-methyl-4-dimethylamidosulphonyl- γ -phenoxybutyric acid (I) and the 4-methyl-2-amidosulphonyl- γ -phenoxy-butyric acid (II).

The electronic absorption spectra of the above derivatives, dissolved in 22 polar solvents have been recorded. The energies of transition of the electronic bands have also been calculated by using the molecular orbital method in Hückel approximation and a model of limited configurational interaction [1].

Since the investigated compounds have significant applications in the synthesis of some drugs, the solvent influence upon the electronic absorption spectra has also to be studied. Thus a study was carried out on the intermolecular interaction forces, manifested in liquid solutions, when the investigated compounds were dissolved in a large variety of solvents with different physico-chemical properties.

According to the theories accounting for the solute-solvent interactions [2]...[4], the modification of pure electronic transition when a substance passes from the vapour state into solution can be written as

$$(1) \quad \Delta\nu_{el} = \Delta\nu_n + \Delta\nu_{ne} = A \frac{n^2 - 1}{2n^2 + 1} + BF(n, \epsilon),$$

where n is the index of refraction and ϵ the electric permittivity of the solution; A and B are constants, depending on the properties of investigated substance.

The function $F(n, \epsilon)$ depends in a very complicated manner on n and ϵ , as well as on some microscopical parameters of the investigated molecule.

The first term in Eq. (1) gives the change of frequencies of pure electronic transitions due to the dispersion intermolecular interactions, and the second term indicate the frequency modifications under influence of intermolecular forces of dipole-dipole type and of the polarization fluctuations in the solutions accomplished in polar solvents.

The influence of medium on the absorption electronic spectra has been studied by following the changes of the position of the electronic band maximum from the long wavelengths, the band α in Platt's notation (I_{Lb}), when the investigated compounds have dissolved in various polar solvents. This could be achieved, since the modification of the maximum of electronic bands is not accompanied by a significant change of the band form.

Since the investigated compounds are not soluble in non-polar or weakly polar solvents, the correlation between the theoretical and experimental data was carried out by taking into account, according to the existing theories accounting for the solute-solvent interactions, only the function $F(n, \epsilon)$ of the form

$$(2) \quad F(n, \epsilon) = B \left(\frac{\epsilon - 1}{\epsilon + 2} - \frac{n^2 - 1}{n^2 + 2} \right).$$

2. Experimental. The molecular electronic spectra of the 4-amido-sulphonyl phenoxyacetic acid derivatives were recorded on a SPECORD UV-VIS Carl Zeiss Jena spectrophotometer. The solution concentrations

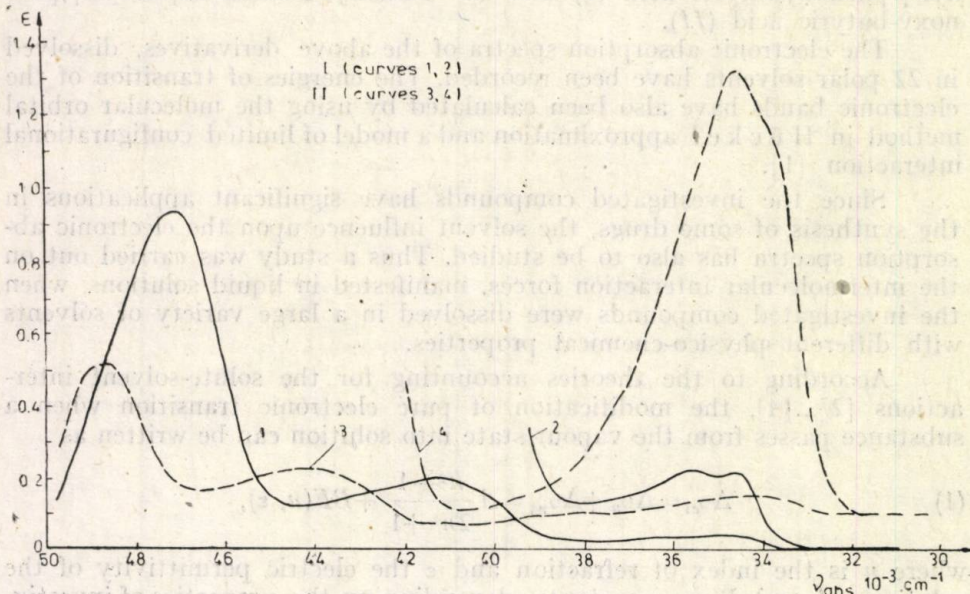


Fig. 1

varied between $3 \cdot 10^{-4}$ and $1 \cdot 10^{-6}$ mole/l and the thicknesses of the absorbing layer between 2 and 10 mm. Under these conditions the extinction values for the absorption electronic band from the long wavelengths were found to lie within the 0.5...0.8 range, where the errors of measurements are minimum.

All of the investigated solvents were either cally pure or analytical grade being provided by the „Reactivul” București, Merck and Carlo Erba firms.

In Fig. 1 the absorption electronic spectra of the investigated compounds dissolved in methanol are showed. It can be seen that both derivatives exhibit three absorption bands, approximately within the same spectral range as the electronic bands of benzene.

In Table 1 the calculated values of the energies of transition are listed and the comparison is made with the experimental results. A good agreement between the calculated and experimental data is found. In this table the positions of the electronic bands correspond to the case when the investigated molecules have been dissolved in methanol.

Table 1

The transition energies — calculated and experimental values

Investigated molecule	Electronic band								
	$\lambda_1(\alpha)$ (eV)			$\lambda_2(\beta)$ (eV)			$\lambda_3(\beta)$ (eV)		
	Experiment	Theory	E	Experiment	Theory	E	Experiment	Theory	E
I	4.29	4.39	0.1	5.02	5.08	0.06	5.85	5.86	0.01
II	4.29	4.41	0.12	5.48	4.82	0.66	6.02	5.89	0.13

In Table 2 the polar solvents used are listed along with their macroscopic characteristics, the index of refraction n for the wavelength $\lambda=4047$ Å, the electrical permittivity ϵ and the positions of the maximum of the investigated absorption electronic band.

The results of correlations between the shifts of maxima of the absorption electronic band from long wavelengths and values of the function $F(n, \epsilon)$ are depicted in Figs. 2—3, where the point numbering coincides with the solvent order given in Table 2.

The linear dependences of the function $F'[\nu_{\max}^{\text{abs}}F(n, \epsilon)]$ in Figs. 2—3 satisfy the equations

$$(3) \quad \nu_{\max}^{\text{abs}}(I) = 35\,440 - 680 F(\epsilon, n),$$

$$(4) \quad \nu_{\max}^{\text{abs}}(II) = 34\,850 - 350 F(\epsilon, n).$$

It can be seen that, in case of the investigated compounds, the positions of the maxima of l_{L_b} bands shifts towards longer wavelengths with

Table 2

No	Solvent	$n_{(4047\text{\AA})}$	ϵ	The position of electronic band (α), cm^{-1}	
				Molecule I	Molecule II
1	Ethylalcohol	1.369	24.0	34 370	34 370
2	Methyl alcohol	1.335	31.0	34 580	34 580
3	Distilled Water	1.342	80.0	34 340	34 340
4	Dioxane	1.426	—	34 680	34 680
5	<i>n</i> -Propyl alcohol	1.393	20.0	34 600	34 600
6	Sec.-Hexyl alcohol	1.425	14.7	34 760	34 760
7	<i>N</i> , <i>N</i> -dimethyl formamide	1.446	37.0	34 560	34 560
8	Dimethyl sulphoxide	1.490	48.0	34 450	34 450
9	2,3-Butandiole	1.442	21.0	34 560	34 560
10	Acetic acid	1.395	21.0	34 640	34 640
11	Acetonitrile	1.351	36.0	35 240	35 240
12	Tetrahydrofurane	1.413	7.4	34 760	34 760
13	Propionic acid	1.394	3.1	34 680	34 680
14	<i>n</i> -Amyl alcohol	1.417	15.0	34 600	34 600
15	Iso-amyl alcohol	1.400	14.7	34 680	34 680
16	Sec.-Buthyl alcohol	1.402	15.4	34 660	34 660
17	Acrylonitrile	1.406	32.0	34 560	34 560
18	1,2 Propanediole	1.430	23.4	34 320	34 320
19	Formamide	1.464	109.0	34 240	34 320
20	Ethyl butanol	1.429	19.7	34 680	34 680
21	Ethyl hexyl alcohol	1.440	6.8	34 720	34 720
22	Diethyl ether	1.452	4.6	34 840	34 840

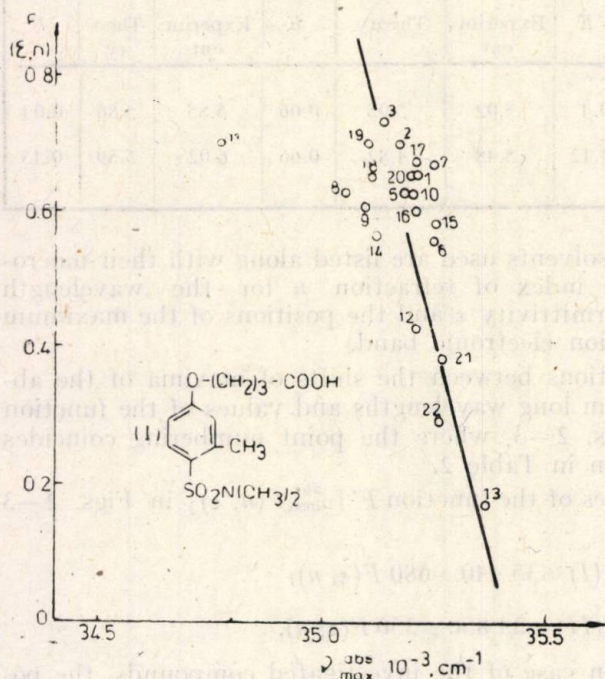


Fig. 2

increasing values of the $F(n, \epsilon)$ function, that is with increasing solvent polarity.

The molecular orbitals calculations and the obtained experimental results allow the assignment of the electronic band from long wavelengths, by means of which the medium influence was studied, to a $\pi - \pi^*$ transition.

The obtained results allow also the conclusion to be drawn that in case of the investigated compounds the changes of the frequencies of the absorption electronic band maximum in polar solvents are due to the orientation and induction interactions (dipole-dipole) as well as to the polarization

fluctuations manifested under these conditions. Hence, we can admit that as regards the medium influence the behaviour of the investigated molecules is in agreement with the theories accounting for the solute-solvent interactions.

It can also be noticed that the changes of maxima of investigated electronic bands are relatively small (the overall shift $\approx 300 \text{ cm}^{-1}$) which indicate that the acids under study exhibit a low sensitivity as regards the medium influence. This behaviour is in full agreement with the wide possibilities of using these substance in the drug industry.

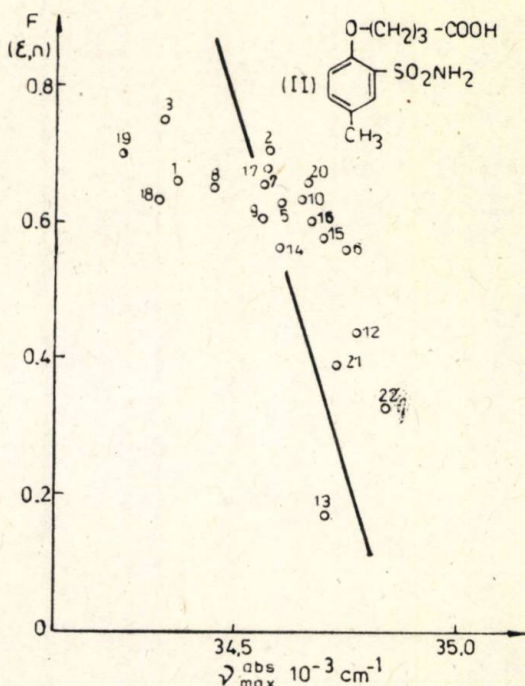


Fig. 3

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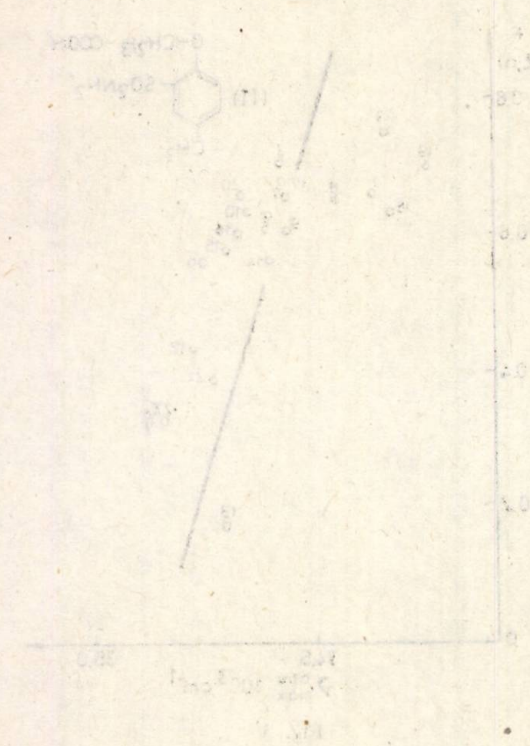
REFERENCES

1. Ghirvu C., Strat G., Strat M., Anal. șt. Univ. Iași, „Al. I. Cuza“ Iași, s. I b, **XXVII**, 45 (1981).
2. Mc. Rae E. G., J. Phys. Chem., **61**, 562 (1957).
3. Баксиев Н. Г., Оптика и спектроскопия, **10**, 717 (1961).
4. Pop V., Anal. șt. Univ. „Al. I. Cuza“ Iași, s. I b, 1279 (1966).

INFLUENȚA SOLVENTULUI ASUPRA SPECTRELOR ELECTRONICE DE ABSORBȚIE ALE UNOR DERIVAȚI AI ACIDULUI 4-AMIDO-SULFONIL-FENOXIACETIC

(Rezumat)

Se studiază spectrele electronice de absorbție ale unor derivați de 4-amido-sulfonil-fenoxiacetic în 22 solvenți polari. S-au determinat prin metoda orbitalilor moleculari pozițiile maximelor benzilor de absorbție ale acestor substanțe.



the relative viscosity η_r of these solutions. Hence, we can admit that as regards the medium influence on the behaviour of the investigated molecules is in agreement with the theories accounting for the solute-solvent interactions. It can also be noticed that the changes of η_r and of investigated $\lg \eta_r$ as a function of the concentration of the acids are relatively small (the overall shift is about 0.1) which indicates that the acids under study exhibit a low sensitivity as regards the medium influence. This behaviour is in full agreement with the wide possibilities of using these substances in the food industry.

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REFERENCES

1. J. Kowalski, *Studia Polym. Sci.*, **11**, 1051 (1973).
2. M. Kowalski, *Studia Polym. Sci.*, **11**, 1051 (1973).
3. J. Kowalski, *Studia Polym. Sci.*, **11**, 1051 (1973).
4. J. Kowalski, *Studia Polym. Sci.*, **11**, 1051 (1973).

THE POLYMERIZATION OF VINYL MONOMERS IN THE PRESENCE OF A CATALYST

The polymerization of vinyl monomers in the presence of a catalyst is a well-known process. The mechanism of this process is still not fully understood. It is generally accepted that the catalyst acts as a free radical initiator. The catalyst is first converted into a free radical, which then initiates the polymerization of the monomer. The rate of polymerization is proportional to the concentration of the catalyst. The molecular weight of the polymer is also affected by the concentration of the catalyst. The polymerization of vinyl monomers in the presence of a catalyst is a complex process, and further research is needed to elucidate the mechanism of this process.

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1. Electronic meter for thermic energy :
 - Turn temperature : $30^{\circ} \dots 150^{\circ}\text{C}$;
 - Return temperature : $25^{\circ} \dots 130^{\circ}\text{C}$;
 - Difference of temperatures : $5^{\circ} \dots 100^{\circ}\text{C}$.
2. Electromagnetic flow sensing device :
 - Rated diameters 10 ... 800 mm ;
 - Rated pressures : 6 ; 10 ; 16 ; 25 bars ;
 - Liquid minimum conductivity : $200 \mu\text{S/m}$;
 - Precision : 1%.
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 - Rated diameters ; 1,5 ... 150 mm ;
 - Operating pressure range : 0 ... 50 bars ;
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 - Rated diameters : 25 ; 32 ; 50 ; 65 ; 80 ; 100 mm ;
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 - Precision : 2.5% (1.5% optional).
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 - Operating range : 600 ... 2 000°C ;
 - Precision : $\pm 1.5\%$ for the domain 600 ... 2 000°C ;
 - Time constant : 0.8 s.
6. Thermal resistances :
 - Measured temperature range : $200^{\circ} \dots 500^{\circ}\text{C}$;
 - Thermoelement material : Pt, Cu ;
 - Design version ; normal, high speed, bend version, for high pressure, flat miniature version, for rooms, for oilers and sternposts.
7. Thermocouples :
 - Measured temperature range : $0^{\circ} \dots 1\,600^{\circ}\text{C}$;
 - Thermocouples material : chromel-alumel, iron-constantan, PtRh (10^{0/0})-Pt, PtBh (30^{0/0})-PtRh (6^{0/0}).
 - Design version : normal, subminiature, high speed, with bend, for high pressures. miniature, for distilleries, multisheats.
8. Direct temperature controllers :
 - Rated diameters : 15 ... 150 mm ;
 - Rated pressures : 16 ; 25 ; 40 bars ;
 - Temperature adjustment range : $0^{\circ} \dots 200^{\circ}\text{C}$.
9. Direct pressure controllers :
 - Rated diameters : 15 ... 150 mm ;
 - Rated pressures : 16 ; 25 ; 40 bars ;
 - Maximum operating temperature : 225°C ;
 - Operating fluid : steam, water, air.
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 - Rated diameters : 50 ... 1 000 mm ;
 - Rated pressures : 25 ... 350 bars ;
 - Maximum operating temperature : 500°C ;
 - Operating fluid : gases, liquids, steam.



CENTRE OF
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JASSY ROMANIA

No. 47, Splai Bahlui
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TESLAMETER FOR ALTERNATING FIELDS TA-1

MEASURING RANGES

The main aim of this device is to measure alternating magnetic induction from plant facilities, apparatus, plant equipments as electric generators, electric motors, electrificated railways and so on. Especially it can measure leakage induction, namely the power wasted in electrical equipments and also the level of magnetic pollution.

PRESENTATION

TA-1 has the following components :

- The electronic block
- A set of detecting elements (4 pieces).

The detecting elements are coils (with air core) with a simple manufacturing.

The electronic block is assembled in a metallic box that is easy to carry. The output is indicated on an analogic display.

The distance between the detecting element and the electronic block that works the signal can be at most 5 m.

TECHNICAL CHARACTERISTICS

Dimensions

— Electronics box	300×245×160 mm
— The set of detecting elements	exterior diameter of the coils $\varnothing 107$ mm ; $\varnothing 38$ mm ; $\varnothing 19$ mm ; $\varnothing 4$ mm

Mass

3.5 kg

Measuring range

0—3 T with subdivisions ; to each subdivision corresponds a special detecting element

The detecting element S_0 0— 3×10^{-5} T

The detecting element S_1 0— 3×10^{-4} T

The detecting element S_2 0— 3×10^{-2} T

The detecting element S_3 0—3 T

When measuring the induction, the precision is 1% of the maximum value of the corresponding subdivision

Relative error

$\epsilon_R \leq 2.5\%$

Voltage

$220 \pm 10\%$ 50 Hz

Temperature range

+5°C—+4°C

Relative humidity

20%—80% without condensation

Working position for the detecting element : the coil axis must be parallel with the magnetic alternating induction.

DELIVERING CONDITIONS

Firm order.



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MILITESLAMETER FOR D.C. FIELDS mTC-1

MEASURING RANGES

The aim of this device is to measure the induction of d.c. or slow changing magnetic fields in laboratory as well as in factories and atmosphere. It can be used for the measuring of coils characteristics or remanent magnetization of some items. This device is also useful in any area of economical activity where we must control and measure the magnetic induction or we must know the demagnetization state of some ferro and ferimagnetic items (it is the case when we fabricate and use anti-friction bearings, crankshafts, some components of an engine etc.).

PRESENTATION

mTC-1 has two main components :

- The electronic block
- The detecting element — a saturated ferrite core flux-gate.

The detecting element is manufactured in a simple manner and does not influence the technological process of the item under control.

TECHNICAL CHARACTERISTICS

Dimensions

- Electronics box $350 \times 240 \times 160$ mm
- Detecting element $\varnothing 21$ mm $l = 125$ mm

Mass 3 kg

Measuring range $0 \pm 3 \times 10^{-3}$ T with 3 subdivisions
 $0 \pm 5 \times 10^{-4}$ T
 $0 \pm 1 \times 10^{-3}$ T
 $0 \pm 2.5 \times 10^{-3}$ T

Precision 1.5%

Relative error $\varepsilon_R < 3\%$

Voltage $220 \text{ V} \pm 10\%$ 50 Hz

Apparent power 15 VA

Temperature range $+5^\circ\text{C} - +40^\circ\text{C}$

Relative humidity $20\% - 80\%$ without condensation

Working position for the detecting element

- for the demagnetization state control: in contact with the item surface
- for measuring of magnetic induction: with its axis being parallel with the magnetic induction

DELIVERING CONDITIONS

Firm order.



Form order.

DELIVERING CONDITIONS

— for measuring of magnetic induction : with its axis being parallel with the magnetic induction

— for the demagnetization state control : in contact with the item surface

Working position for the detecting element

Relative humidity 30%—80% without condensation

Temperature range $+5^{\circ}\text{C}$ — $+40^{\circ}\text{C}$

Apparent power 15 VA

Voltage $220\text{ V} \pm 10\%$ 50 Hz

Relative error $\leq 3\%$

Precision 1.5%

$0 \pm 2.5 \times 10^{-4}\text{ T}$

$0 \pm 1 \times 10^{-4}\text{ T}$

$0 \pm 5 \times 10^{-4}\text{ T}$

Measuring range $0 \pm 3 \times 10^{-4}\text{ T}$ with 3 subdivisions

Mass 3 kg

— Detecting element 231 mm $l=125$ mm

— Electronics box 350 $\times$ 240 $\times$ 160 mm

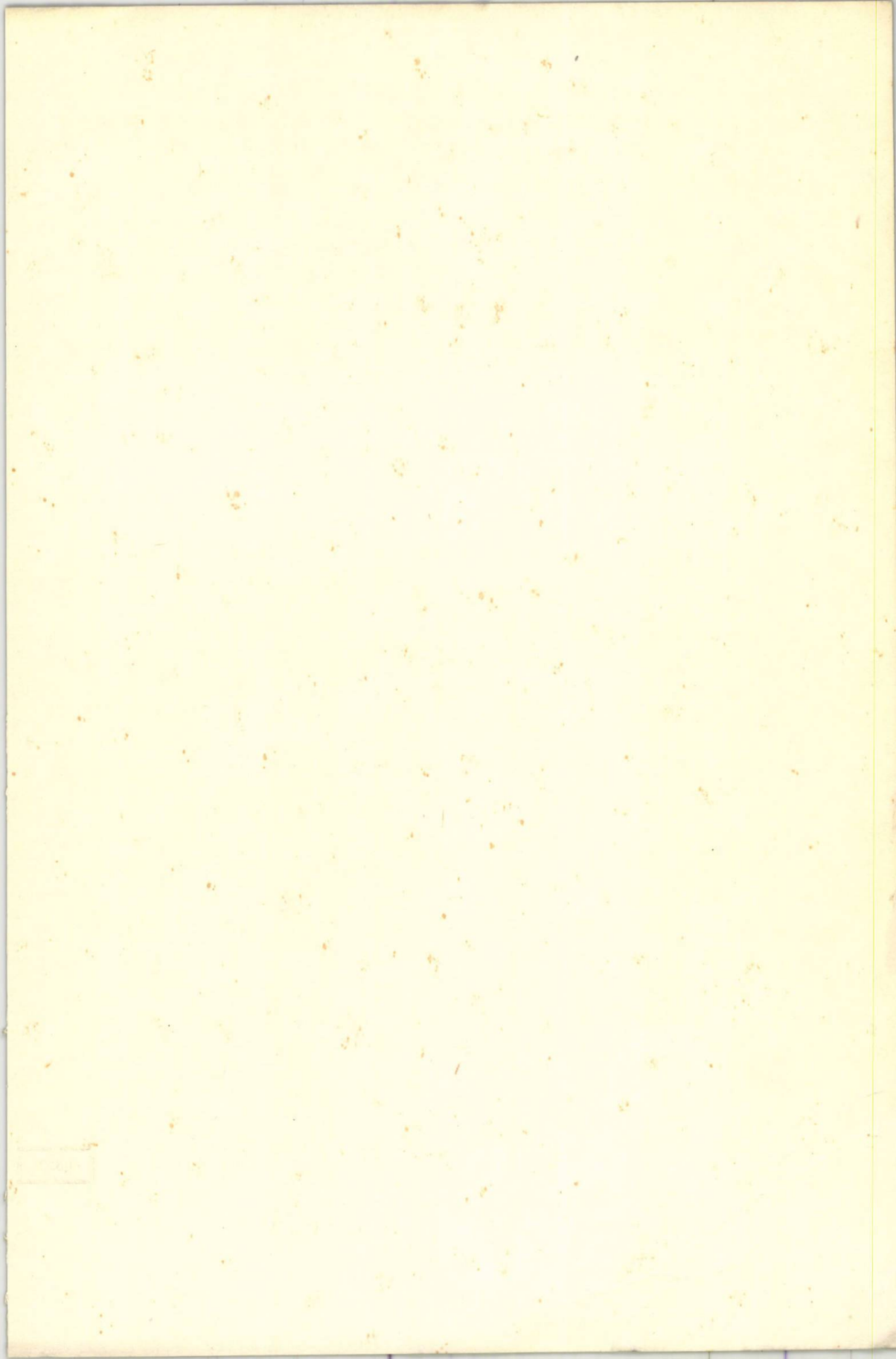
Dimensions

TECHNICAL CHARACTERISTICS



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